Note Title 4/19/2011

Stability

$y' = \lambda y, \quad y(0) = 1$

Exact Solution : $y(x) = e^{\lambda x}, \quad x \in [0, b]$

$h = \frac{b}{N}, \quad x_n = nh$

Euler's Method : $y_{n+1} = y_n + h\lambda y_n$

$$= (1 + h\lambda) y_n$$

$$= (1 + h\lambda)^{n+1}$$

Rungé – Kutta Method

$$y' = \lambda y \,, \quad y(0) = 1 \,,$$

$$y_{n+1} = y_n + \frac{k_1 + k_2}{2} \,,$$

$$k_1 = h\, f(x_n, y_n) = h \lambda y_n$$

$$k_2 = h\, f(x_n + h, y_n + k_1) = h \lambda (1 + h\lambda) y_n$$

$$y_{n+1} = y_n + h \lambda y_n + \frac{h^2 \lambda^2}{2} y_n$$

$$= \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^{n+1}$$

$$e^{x} = 1 + x + \frac{x^2}{2} e^{c}$$
$$= 1 + x + O(x^2)$$

$$e^{nx} = (1+x)^n + O(x^2)$$

$$e^{nh\lambda} = (1 + h\lambda)^n + O(h^2\lambda^2)$$

Euler's Method

$$y(x_n) - y_n = O(h^2\lambda^2).$$

$$e^{x} = 1 + x + \frac{x^2}{2} + O(x^3)$$

$$e^{nx} = \left(1 + x + \frac{x^2}{2}\right)^n + O(x^3)$$

$$e^{nh\lambda} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n + O(h^3\lambda^3)$$

<u>Rungé</u> – <u>Kutta</u> <u>Method</u>

$$y(x_n) - y_n = O(h^3\lambda^3)$$

Midpoint Method

$y' = \lambda y \,, \quad y(0) = 1$

$$y_{n+1} = y_{n-1} + 2h f(x_n, y_n)$$
$$= y_{n-1} + 2h\lambda y_n$$

Difference Equation:

$$y_{n+1} - 2h\lambda y_n - y_{n-1} = 0$$

$y_0 = 1 \,, \quad y_1 = y(h) = e^{\lambda h}$

$$y_{n+1} - 2h\lambda y_n - y_{n-1} = 0 \ , \ n = 1, 2, \cdots$$

$$y_n = r^n$$

$$r^{n+1} - 2h\lambda r^n - r^{n-1} = 0 \implies$$

$$r^2 - 2h\lambda - 1 = 0$$

$$r_1 = \frac{2h\lambda + \sqrt{4h^2\lambda^2 + 4}}{2} = h\lambda + \sqrt{1 + h^2\lambda^2}$$

$$r_2 = h\lambda - \sqrt{1 + h^2\lambda^2}$$

$$y_n = \beta_1 r_1^n + \beta_2 r_2^n \ , \ y_0 = 1 \ , \ y_1 = e^{h\lambda}$$

$$\ddot{\ }$$

$$y_n = \beta_1 r_1^n + \beta_2 r_2^n$$

$$y_0 = 1, \quad y_1 = e^{h\lambda}$$

$$r_1 = h\lambda + \sqrt{1+h^2\lambda^2}, \quad r_2 = h\lambda - \sqrt{1+h^2\lambda^2}$$

$$\beta_1 + \beta_2 = 1, \quad \beta_1 r_1 + \beta_2 r_2 = y_1$$

$$\beta_2 = 1 - \beta_1, \quad \beta_1(r_1 - r_2) + r_2 = y_1$$

$$\beta_1 = \frac{y_1 - r_2}{2\sqrt{1+h^2\lambda^2}} \longrightarrow 1 \quad \text{as } h \longrightarrow 0$$

$$\beta_2 \longrightarrow 0$$

$$y' = \lambda y \,, \quad y(0) = 1$$

$$y_n = \beta_1 r_1^n + \beta_2 r_2^n \,,$$

$$r_1 = h\lambda + \sqrt{1 + h^2\lambda^2} \,, \quad r_2 = h\lambda - \sqrt{1+h^2\lambda^2}$$

$$f(x) = \sqrt{1+x} \,, \quad f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$f(x) = 1 + x\, f'(c) = 1 + O(x)$$

$$r_1 = h\lambda + \sqrt{1+h^2\lambda^2}$$
$$= h\lambda + 1 + O(h^2\lambda^2)$$
$$r_2 = h\lambda - \sqrt{1+h^2\lambda^2}$$
$$= h\lambda - 1 + O(h^2\lambda^2)$$
$$r_1^n = (1+h\lambda)^n + O(h^2\lambda^2)$$
$$r_2^n = (-1)^n(1-h\lambda)^n + O(h^2\lambda^2)$$

$$
\begin{aligned}
y_n &= \beta_1 r_1^n + \beta_2 r_2^n \\
&= \beta_1 (1 + h\lambda)^n + \beta_2 (-1)^n (1 - h\lambda)^n \\
&\qquad + O(h^2 \lambda^2) \\
&= \beta_1 e^{nh\lambda} + \beta_2 (-1)^n e^{-nh\lambda} \\
&\qquad + O(h^2 \lambda^2) \\
&= \beta_1 e^{\lambda x_n} + \beta_2 (-1)^n e^{-\lambda x_n} \\
&\qquad + O(h^2 \lambda^2)
\end{aligned}
$$

$$y' = \lambda y, \quad y(0) = 1$$

$$y(x) = e^{\lambda x}$$

$$y_n = \beta_1 e^{\lambda x_n} + \beta_2 (-1)^n e^{-\lambda x_n} + O(h^2 \lambda^2)$$

$\beta_1 \to 1, \quad \beta_2 \to 0 \quad \text{as} \quad h \to 0.$

$$y_n = \beta_1 e^{\lambda x_n} + \underbrace{\beta_2 (-1)^n e^{-\lambda x_n}}_{\text{extraneous solution}} + O(h^2 \lambda^2)$$

$\lambda > 0$: OK!

$\lambda < 0$: extraneous solution dominates the true solution

error increases exponentially.

Adams-Bashforth Method

$$y_{n+1} = y_n + \frac{h}{24}\left[55 g_n - 59 g_{n-1} + 37 g_{n-2} - 9 g_{n-3}\right]$$

$$g_n = g(x_n) = f(x_n, y_n)$$

$$y_n = \underbrace{\beta_1 r_1^n}_{\text{true solution}} + \beta_2 r_2^n + \beta_3 r_3^n + \beta_4 r_4^n$$

$|r_2| < 1$, $|r_3| < 1$, $|r_4| < 1$, h small enough

$y' = \lambda y$, $y(0) = 1$: Stable

Boundary Value Problem

$$y''(x) + f(x)\,y'(x) + g(x)\,y(x) = r(x),$$

$$x \in [a, b]$$

$$y(a) = \alpha, \quad y(b) = \beta$$

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}$$

f, g, r : continuous.

$$a = x_0 < x_1 < \cdots < x_N = b$$

$$h = \frac{b-a}{N}$$

<u>Interior</u> <u>mesh</u> <u>points</u>

$$x_n = x_0 + nh \; : \; n = 1, 2, \cdots, N-1$$

$$y''(x) + f(x)\,y'(x) + g(x)\,y(x)$$
$$= r(x), \quad x \in [a,b]$$

$$y(a) = \alpha, \quad y(b) = \beta.$$

$$x_n = a + nh, \quad n = 0, 1, \cdots, N$$

$$y''(x_n) + f(x_n)\,y'(x_n) + g(x_n)\,y(x_n)$$
$$= r(x_n), \quad n = 1, \cdots, N-1$$

$$y''(x_n) + f(x_n)\,y'(x_n) + g(x_n)\,y(x_n)$$
$$= r(x_n), \quad n = 1, \cdots, N-1$$

$$y(x_n) \simeq y_n$$

$$y'(x_n) \simeq \frac{y_{n+1} - y_{n-1}}{2h}$$

$$y''(x_n) \simeq \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} + f(x_n)\frac{y_{n+1} - y_{n-1}}{2h}$$

$$+ g(x_n)\, y_n = r(x_n),$$

$$n = 1, 2, \cdots, N-1$$

$$\left(1 - \frac{h}{2} f_n\right) y_{n-1} + \left(-2 + h^2 g_n\right) y_n$$

$$+ \left(1 + \frac{h}{2} f_n\right) y_{n+1} = h^2 r_n,$$

$$n = 1, 2, \cdots, N-1$$

$$\left(1-\frac{h}{2}f_n\right)y_{n-1}+\left(-2+h^2g_n\right)y_n$$
$$+\left(1+\frac{h}{2}f_n\right)y_{n+1}=h^2r_n,$$
$$n=1,2,\cdots,N-1$$

$$a_n\,y_{n-1}+d_n\,y_n+c_n\,y_{n+1}=h^2r_n$$
$$n=1,2,\cdots,N-1$$

$n=1$

$$d_1\,y_1+c_1\,y_2=h^2r_1-a_1\alpha$$

$n=N-1$

$$a_{N-1}\,y_{N-2}+d_{N-1}\,y_{N-1}=h^2r_{N-1}-c_{N-1}\beta$$

$$d_1 y_1 + c_1 y_2 = h^2 r_1 - a_1 \alpha$$

$$a_n y_{n-1} + d_n y_n + c_n y_{n+1} = h^2 r_n$$

$$n = 2, 3, \cdots, N-2$$

$$a_{N-1} y_{N-2} + d_{N-1} y_{N-1} = h^2 r_{N-1} - c_{N-1} \beta$$

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_2 & d_2 & c_2 & & \\ & a_3 & d_3 & c_3 & \\ & & & \ddots & \\ & & & a_{N-1} & d_{N-1} \end{bmatrix}$$

Tridiagonal matrix.

Recall :

$$f(x) = f(a-h) + f[a-h \;\; a+h](x-a+h)$$

$$+ f[a-h \;\; a+h \;\; x](x-a+h)(x-a-h)$$

$$f'(x) = f[a-h \;\; a+h] +$$

$$f[a-h \;\; a+h \;\; x \;\; x](x-a+h)(x-a-h)$$

$$+ f[a-h \;\; a+h \;\; x]\{x-a-h + x-a+h\}$$

$$f'(a) = f[a-h \;\; a+h] - h^2 f[a-h \;\; a+h \;\; a \;\; a]$$

$$f'(a) \simeq \frac{f(a+h) - f(a-h)}{2h}, \quad \text{error} = \frac{f^{(3)}(c)}{3!} h^2$$

$$A \bar{x} = \bar{b}$$

$$A = \begin{bmatrix} d_1 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & d_2 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & d_3 & c_3 & 0 \cdots & 0 & 0 \\ & & & & & & \\ 0 & 0 & \cdots & 0 & a_{N-2} & d_{N-2} & c_{N-2} \\ 0 & 0 & \cdots & 0 & & a_{N-1} & d_{N-1} \end{bmatrix}$$

Example : $y'' + y = 0$, $y(0) = 0$, $y(1) = 1$.

Exact solution : $y(x) = \dfrac{\sin x}{\sin 1}$.

$f(x) \equiv 0$, $g(x) \equiv 1$, $q(x) \equiv 0$

$d_n = h^2 g_n - 2 = h^2 - 2$, $a_n = c_n = 1$.