Note Title 4/21/2011

Complex Vectors

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n, \quad z_i \in \mathbb{C}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$$

$$z + w = \begin{bmatrix} z_1 + w_1 \\ z_2 + w_2 \\ \vdots \\ z_n + w_n \end{bmatrix}, \quad \alpha z = \begin{bmatrix} \alpha z_1 \\ \alpha z_2 \\ \vdots \\ \alpha z_n \end{bmatrix}, \quad \alpha \in \mathbb{C}$$

Inner Product

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i},$$

$\overline{w_i}$: complex conjugate

$$\langle z, z \rangle = \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2$$

$$\langle z, z \rangle \geq 0, \quad \langle z, z \rangle = 0 \iff z = \overline{0}$$

$$\langle w, z \rangle = \sum_{i=1}^{n} w_i \overline{z_i} = \overline{\left(\sum_{i=1}^{n} z_i \overline{w_i} \right)}$$

$$= \overline{\langle z, w \rangle}$$

$$\langle z+v, w\rangle = \sum_{i=1}^{n} (z_i + v_i)\overline{w_i}$$

$$= \sum_{i=1}^{n} z_i \overline{w_i} + \sum_{i=1}^{n} v_i \overline{w_i}$$

$$= \langle z, w\rangle + \langle v, w\rangle$$

$$\langle \alpha z, w\rangle = \sum_{i=1}^{n} \alpha z_i \overline{w_i}$$

$$= \alpha \sum_{i=1}^{n} z_i \overline{w_i} = \alpha \langle z, w\rangle$$

Properties of the inner product:

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

1) $\langle z, z \rangle \geq 0$, $\langle z, z \rangle = 0 \iff z = \overline{0}$

2) $\langle w, z \rangle = \overline{\langle z, w \rangle}$

3) $\langle z + v, w \rangle = \langle z, w \rangle + \langle v, w \rangle$

$\langle \alpha z, w \rangle = \alpha \langle z, w \rangle$

$\langle z, \alpha w \rangle = \overline{\alpha} \langle z, w \rangle$

$$\langle z, z \rangle = \sum_{i=1}^{n} |z_i|^2$$

$$\|z\|_2 = \sqrt{\langle z, z \rangle}$$ Induced norm

<u>Cauchy - Schwarz Inequality:</u>

$$|\langle z, w \rangle| \leq \|z\|_2 \, \|w\|_2$$

Norm

$$\|z\|_2 = \sqrt{\sum_{i=1}^{n} |z_i|^2}$$

1) $\|z\|_2 \geq 0 \,, \quad \|z\|_2 = 0 \iff z = \overline{0}$

2) $\|\alpha z\|_2 = |\alpha| \, \|z\|_2$

3) $\|z + w\|_2 \leq \|z\|_2 + \|w\|_2$

$$\|z\|_2 = \sqrt{\sum_{i=1}^{n} |z_i|^2}$$

$$\|z\|_1 = \sum_{i=1}^{n} |z_i|$$

$$\|z\|_\infty = \max_{1 \leq i \leq n} |z_i|$$

$A = [a_{ij}]$: $n \times n$ complex matrix

Induced Matrix Norm

$$\|A\| = \max_{z \neq \overline{0}} \frac{\|Az\|}{\|z\|}$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad : \text{Column-sum norm}$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad : \text{Row-Sum norm}$$

Frobenius Norm

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$\|A\|_2$: not computable.

$$\|A\|_F \leq \|A\|_2$$

Basic Inequality

$$\|A\| = \max_{z \neq \bar{0}} \frac{\|Az\|}{\|z\|}$$

$$\boxed{\|Az\| \leq \|A\| \, \|z\|, \quad z \in \mathbb{C}^n}$$

Conjugate-transpose

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad z^* = \bar{z}^t$$

$$= [\bar{z}_1 \ \bar{z}_2 \ \cdots \ \bar{z}_n]$$

$$\langle z, \omega \rangle = \sum_{i=1}^{n} z_i \overline{\omega_i}$$

$$= \omega^* z$$

A : real / complex $n \times n$ matrix

$A^* = \overline{A}^t$: Conjugate - transpose

$(A^*)^* = A$

$(AB)^* = (\overline{AB})^t = (\overline{A}\,\overline{B})^t = \overline{B}^t\,\overline{A}^t$

$= B^* A^*$

$\langle Az, w \rangle = w^* A z = (A^* w)^* z$

$= \langle z, A^* w \rangle$

Special Matrices

A normal : $A^* A = A A^*$

A self-adjoint : $A^* = A$

A skew self-adjoint : $A^* = -A$

A unitary : $A^* A = I \; (= A A^*)$

Eigenvalue Problem

A : $n \times n$ real / complex matrix

Definition: A complex number λ is said to be an eigenvalue of A if there exists a non-zero vector u such that

$$Au = \lambda u.$$

u : associated eigenvector

$$Au = \lambda u, \quad u \neq \overline{0}$$

$\Rightarrow$

$$(A - \lambda I) u = \overline{0}$$

$\Rightarrow$ $A - \lambda I : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is not 1-1

$\Rightarrow$ $A - \lambda I$ is not invertible

$\Rightarrow$ $\det(A - \lambda I) = 0$

Let $\lambda \in \mathbb{C}$ be such that $\det(A - \lambda I) = 0$.

Consider the homogeneous system

$$(A - \lambda I) z = \overline{0}$$

It has a non-trivial solution u.

$$(A - \lambda I) u = \overline{0}, \quad u \neq \overline{0},$$

that is, $Au = \lambda u$, $u \neq \overline{0}$

The eigenvalues of A are given by $\det(A-\lambda I)=0$.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$(-1)^n \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0 = 0$$

Characteristic Polynomial

$$\det(A - \lambda I)$$
$$= (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

It has n roots, counted according to their multiplicities.

Consequence of the Fundamental Theorem of Algebra

$$\det(A-\lambda I)$$

$$= (-1)^n \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0$$

$$= (\lambda_1-\lambda)^{m_1}(\lambda_2-\lambda)^{m_2}\cdots(\lambda_k-\lambda)^{m_k}$$

$$m_1 + m_2 + \cdots + m_k = n$$

The roots of $\det(A-\lambda I)$ are the eigenvalues of A:

$\lambda_1, \lambda_2, \cdots, \lambda_k$

m_i : <u>algebraic multiplicity</u> of λ_i

$$Au = \lambda u, \quad u \neq \overline{0}$$

λ : eigenvalue, u : eigenvector

$$A(\alpha u) = \alpha Au = \alpha(\lambda u)$$
$$= \lambda(\alpha u)$$

u : eigenvector $\Rightarrow$ αu : eigenvector if $\alpha \neq 0$

Eigenspace :

$$N(A-\lambda I) = \{ z \in \mathbb{C}^n : (A-\lambda I)z = \bar{0} \}$$

The eigenspace consists of eigenvectors and the zero vector.

The dimension of $N(A-\lambda I)$ is called the geometric multiplicity of λ.

geometric multiplicity g of λ:

Number of linearly independent eigenvectors associated with λ.

geometric multiplicity $\leq$ algebraic multiplicity

Example:

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\det(A - \lambda I) = (1-\lambda)^2$

1 : eigenvalue of A with algebraic multiplicity 2.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{aligned} u_1 + u_2 &= u_1 \\ u_2 &= u_2 \end{aligned}$$

$$\Rightarrow u_2 = 0$$

$N(A - I) = \left\{ \begin{bmatrix} u_1 \\ 0 \end{bmatrix} : u_1 \in \mathbb{C} \right\}$: geometric multiplicity : 1

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\det(A - \lambda I) = (1-\lambda)(2-\lambda)$

eigenvalues : 1 and 2 with

algebraic multiplicities = 1 .

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow u_1 + u_2 = u_1 , \; 2u_2 = u_2$

$\Rightarrow \begin{bmatrix} u_1 \\ 0 \end{bmatrix} : \overrightarrow{ev} , \; u_1 \neq 0$

geometric multiplicity = 1

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \quad u_1 + u_2 = 2 u_1$$

$$2 u_2 = 2 u_2$$

geometric multiplicity

$= 1$

$$\Rightarrow \quad u_1 = u_2$$

$\vec{ev}$: $\begin{bmatrix} u_1 \\ u_1 \end{bmatrix}, u_1 \neq 0$

$$Au = \lambda u, \quad u \neq \bar{0}, \lambda \in \mathbb{C}$$

$$u^* A u = u^* (\lambda u) = \lambda (u^* u)$$

$$u^* u = \sum_{i=1}^{n} u_i \overline{u_i} = \sum_{i=1}^{n} |u_i|^2 \neq 0 .$$

$$\lambda = \frac{u^* A u}{u^* u} = \frac{\langle Au, u \rangle}{\langle u, u \rangle}$$

$$A u = \lambda u, \quad u \neq \overline{0}$$

$$\lambda = \frac{\langle Au, u \rangle}{\langle u, u \rangle} = \frac{\langle u, A^* u \rangle}{\langle u, u \rangle}$$

$$\overline{\lambda} = \frac{\overline{\langle Au, u \rangle}}{\overline{\langle u, u \rangle}} = \frac{\langle u, Au \rangle}{\langle u, u \rangle}$$

$A^* = A \Rightarrow \overline{\lambda} = \lambda \Rightarrow \lambda$: real

$A^* = -A \Rightarrow \overline{\lambda} = -\lambda \Rightarrow \lambda$: 0 or purely imaginary

A normal : $A^* A = A A^*$

$$\|Ax\|_2^2 = \langle Ax, Ax \rangle$$
$$= \langle x, A^* A x \rangle$$
$$= \langle x, A A^* x \rangle$$
$$= \langle A^* x, A^* x \rangle = \|A^* x\|_2^2$$

$$\boxed{A \text{ normal} \Rightarrow \|Ax\|_2 = \|A^* x\|_2}$$

A normal : $A^*A = AA^*$

$Au = \lambda u, \quad u \neq \overline{0}$

$$\underset{\substack{\| \\ 0}}{\|(A-\lambda I)u\|_2} = \|(A-\lambda I)^* u\|_2$$

$\Rightarrow \quad A^* u = \overline{\lambda} u, \quad u \neq \overline{0}$

$\overline{\lambda}$ is the eigenvalue of A,

same eigenvector