Note Title 4/28/2011

$A$ : $n \times n$ real / complex matrix

$Au = \lambda u, \quad \lambda \in \mathbb{C}, \quad \bar{0} \neq u \in \mathbb{C}^n$

$\lambda$ : eigenvalue , $u$ : eigenvector

$A^* = \bar{A}^t$ : Conjugate transpose

Self-adjoint : $A^* = A$ : real eigenvalues

Skew self-adjoint : $A^* = -A$, eigenvalues : purely imaginary or zero

Normal : $A^* A = A A^*$

$$\|Ax\|_2 = \|A^* x\|_2$$

A normal:

$$Au = \lambda u \iff A^* u = \bar{\lambda} u$$

A: unitary, $A^* A = A A^* = I$

$$Au = \lambda u, \quad u \neq \bar{0}$$

$$\Rightarrow A^* A u = \lambda A^* u = \lambda \bar{\lambda} u$$

$$\Rightarrow u = |\lambda|^2 u \Rightarrow |\lambda| = 1$$

A normal : $A^*A = AA^*$

$Au = \lambda u$, $Av = \mu v$,

$\lambda \neq \mu$, $u \neq \overline{0}$, $v \neq \overline{0}$

$\Rightarrow A^* v = \overline{\mu} v$.

Consider

$\lambda \langle u, v \rangle = \langle \lambda u, v \rangle$

$= \langle Au, v \rangle = \langle u, A^* v \rangle$

$= \langle u, \overline{\mu} v \rangle = \mu \langle u, v \rangle \Rightarrow \underset{= 0}{\langle u, v \rangle}$

**Definition**: Two matrices $A$ and $B$ are said to be similar if there exists an invertible matrix $P$ such that

$$B = P^{-1} A P$$

Similar matrices have the same eigenvalues:

Let $B = P^{-1} A P$. Then

$\det(B - \lambda I) = \det(P^{-1} A P - \lambda I)$

$= \det(P^{-1}(A - \lambda I) P)$

$= \det(P^{-1}) \det(A - \lambda I) \det(P)$

$= \det(P^{-1} P) \det(A - \lambda I) = \det(A - \lambda I)$

$B = P^{-1} A P$

$\det(B - \lambda I) = \det(A - \lambda I)$

Same Characteristic polynomial

$\det(A - \lambda I) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}$

$m_1 + \cdots + m_k = n$

$m_1$ : algebraic multiplicity of $\lambda_1$

Algebraic multiplicities are preserved.

$B = P^{-1} A P$

$Bu = \lambda u, \quad u \neq \overline{0}$

$\iff P^{-1} A P u = \lambda u$

$\iff A(Pu) = \lambda (Pu)$

$u \neq \overline{0} \iff Pu \neq \overline{0}$

$u$ eigenvector of $B$

$\iff Pu$ eigenvector of $A$

$u_1, u_2, \cdots, u_j$ linearly independent

$\iff Pu_1, Pu_2, \cdots, Pu_j$ : lin. indep.

Let $u_1, u_2, \cdots, u_j$ be lin. indep.

Consider

$$\sum_{k=1}^{j} \alpha_k P u_k = \bar{0} \implies P^{-1}\left(\sum_{k=1}^{j} \alpha_k P u_k\right) = \bar{0}$$

$$\implies \sum_{k=1}^{j} \alpha_k u_k = \bar{0} \implies \alpha_k = 0, \quad k = 1, \cdots, j$$

$$B = P^{-1} A P$$

Geometric multiplicity of $\lambda$
= number of linearly independent eigenvectors

Geometric multiplicities are preserved

**Definition:** A matrix $A$ is said to be **diagonalizable** if there exists an invertible matrix $P$ and a diagonal matrix $D$ such that

$$P^{-1} A P = D.$$

$$P^{-1} A P = D = \begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix}$$

$$\iff A P = P D$$

$$\iff A \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} = P D$$

$$A \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} =$$

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix}$$

$$\Longleftrightarrow \quad A p_j = d_{jj} \, p_j \, , \quad j = 1, \cdots, n$$

$d_{jj}$ : eigenvalues , $p_j$ : eigenvectors

$A$ is diagonalizable

$\iff$ there are $n$ linearly independent eigenvectors of $A$

$\iff$ eigenvectors of $A$ form a basis of $\mathbb{C}^n$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1 : eigenvalue

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ : eigenvector

A is <u>not</u> diagonalizable.

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

As a consequence, if A has n distinct eigenvalues, then it is diagonalizable.

## Schur's Theorem

Let $A$ be an $n \times n$ real/complex matrix. Then there exists a unitary matrix $U$ and an upper triangular matrix $T$ such that

$$U^* A U = T$$

## Proof by Induction

$$U^* A U = T, \quad U^* U = U U^* = I,$$

$T$: upper triangular.

The diagonal entries of $T$ are the eigenvalues of $A$.

$U^* A U = T \implies$

$(U^* A U)^* = U^* A^* U = T^*.$

$A^* = A \implies T^* = T$

$T$ : upper triangular

$\implies T$ : real diagonal matrix

$A^* = A \implies A$ : diagonalizable

$U^*AU = T$ , $U^*A^*U = T^*$

$A^* = -A \Rightarrow T^* = -T$

$\Rightarrow T$ : diagonal matrix

diagonal entries : zero or purely imaginary.

$A^* = -A$ : $A$ is diagonalizable

$U^* A U = T, \quad U^* A^* U = T^*$

$U^* U = U U^* = I$ , $T$ : upper triangular

$$\begin{aligned} T T^* &= U^* A U U^* A^* U \\ &= U^* A A^* U \\ &= U^* A U U^* A^* U \\ &= T T^* \end{aligned}$$

$\Rightarrow T = D$ Normal Matrices are diagonalizable

## Spectral Theorem

$A$ normal : $A^*A = AA^*$

There exists a unitary matrix $U$ and a diagonal matrix $D$ such that $U^*AU = D$

$$U^* U = I$$

$$U = [u_1 \quad u_2 \quad \cdots \quad u_n]$$

$$U^* = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix} \qquad U^* U = \begin{bmatrix} u_1^* u_1 & u_1^* u_2 & \cdots & u_1^* u_n \\ u_2^* u_1 & u_2^* u_2 & \cdots & u_2^* u_n \\ \vdots & & & \\ u_n^* u_1 & u_n^* u_2 & \cdots & u_n^* u_n \end{bmatrix}$$

$$= [u_i^* u_j] = [\langle u_j, u_i \rangle]$$

$$U^* U = [\langle u_j, u_i \rangle] = I$$

$$\Rightarrow \langle u_j, u_i \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The columns of $U$ are orthonormal

$$U^* A U = D \Rightarrow AU = UD$$

The columns of $U$ are eigenvectors.

## Spectral Theorem

$A$ : normal. $A$ has $n$ orthonormal vectors. $A u_j = \lambda_j u_j$, $j = 1, \cdots, n$

$$z \in \mathbb{C}^n : z = \sum_{j=1}^{n} \alpha_j u_j$$

$$\langle z, u_k \rangle = \langle \sum_{j=1}^{n} \alpha_j u_j , u_k \rangle$$

$$= \sum_{j=1}^{n} \alpha_j \langle u_j, u_k \rangle = \alpha_k$$

$A$ normal, $\quad A u_j = \lambda_j u_j$, $j = 1, \cdots, n$,

$\|u_j\|_2 = \sqrt{\langle u_j, u_j \rangle} = 1$,

Let $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$

$z \in \mathbb{C}^n \implies z = \sum_{j=1}^{n} \langle z, u_j \rangle u_j$

$$A z = \sum_{j=1}^{n} \langle z, u_j \rangle A u_j$$

$$= \sum_{j=1}^{n} \langle z, u_j \rangle \lambda_j u_j$$

$$z = \sum_{j=1}^{n} \langle z, u_j \rangle u_j$$

$$\langle z, z \rangle = \left\langle \sum_{j=1}^{n} \langle z, u_j \rangle u_j , \sum_{k=1}^{n} \langle z, u_k \rangle u_k \right\rangle$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle z, u_j \rangle \overline{\langle z, u_k \rangle} \langle u_j, u_k \rangle$$

$$= \sum_{j=1}^{n} \left| \langle z, u_j \rangle \right|^2 = \| z \|_2^2$$

$$A z = \sum_{j=1}^{n} \langle z, u_j \rangle \lambda_j u_j$$

$$\| A z \|_2^2 = \sum_{j=1}^{n} |\langle z, u_j \rangle|^2 |\lambda_j|^2$$

$$\leq |\lambda_1|^2 \sum_{j=1}^{n} |\langle z, u_j \rangle|^2$$

$$= |\lambda_1|^2 \| z \|_2^2$$

$$\|Az\|_2 \leq |\lambda_1| \, \|z\|_2$$

$$\Rightarrow \frac{\|Az\|_2}{\|z\|_2} \leq |\lambda_1| \, , \; z \neq \bar{0}$$

$$\Rightarrow \|A\|_2 = \max_{z \neq \bar{0}} \frac{\|Az\|_2}{\|z\|_2} \leq |\lambda_1|$$

$$A u_1 = \lambda_1 u_1 \Rightarrow \|A u_1\|_2 = |\lambda_1| \, \|u_1\|_2$$

$$\Rightarrow |\lambda_1| \leq \|A u_1\|_2 \leq \|A\|_2 \, \|u_1\|_2 = \|A\|_2$$

$A$ normal

$\lambda_1, \cdots, \lambda_n$ : eigenvalues of $A$

$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

$$\boxed{\|A\|_2 = |\lambda_1|}$$