Note Title 4/28/2011

Spectral Theorem

A : normal. A has n orthonormal vectors. $Au_j = \lambda_j u_j$, $j = 1, \cdots, n$

$$z \in \mathbb{C}^n : \quad z = \sum_{j=1}^{n} \alpha_j u_j$$

$$\langle z, u_k \rangle = \langle \sum_{j=1}^{n} \alpha_j u_j, u_k \rangle$$

$$= \sum_{j=1}^{n} \alpha_j \langle u_j, u_k \rangle = \alpha_k$$

A normal, $Au_j = \lambda_j u_j$, $j = 1, \cdots, n$,

$\| u_j \|_2 = \sqrt{\langle u_j, u_j \rangle} = 1$,

Let $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

$z \in \mathbb{C}^n \implies z = \sum_{j=1}^{n} \langle z, u_j \rangle u_j$

$$Az = \sum_{j=1}^{n} \langle z, u_j \rangle A u_j$$

$$= \sum_{j=1}^{n} \langle z, u_j \rangle \lambda_j u_j$$

$$z = \sum_{j=1}^{n} \langle z, u_j \rangle u_j$$

$$\langle z, z \rangle = \left\langle \sum_{j=1}^{n} \langle z, u_j \rangle u_j , \sum_{k=1}^{n} \langle z, u_k \rangle u_k \right\rangle$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle z, u_j \rangle \overline{\langle z, u_k \rangle} \langle u_j , u_k \rangle$$

$$= \sum_{j=1}^{n} | \langle z, u_j \rangle |^2 = \| z \|_2^2$$

$$A z = \sum_{j=1}^{n} \langle z, u_j \rangle \lambda_j u_j$$

$$\| A z \|_2^2 = \sum_{j=1}^{n} | \langle z, u_j \rangle |^2 | \lambda_j |^2$$

$$\leq | \lambda_1 |^2 \sum_{j=1}^{n} | \langle z, u_j \rangle |^2$$

$$= | \lambda_1 |^2 \| z \|_2^2$$

$$\|Az\|_2 \leq |\lambda_1| \, \|z\|_2$$

$$\Rightarrow \frac{\|Az\|_2}{\|z\|_2} \leq |\lambda_1| \, , \; z \neq \bar{0}$$

$$\Rightarrow \|A\|_2 = \max_{z \neq \bar{0}} \frac{\|Az\|_2}{\|z\|_2} \leq |\lambda_1|$$

$$A u_1 = \lambda_1 u_1 \Rightarrow \|A u_1\|_2 = |\lambda_1| \, \|u_1\|_2$$

$$\Rightarrow |\lambda_1| \leq \|A u_1\|_2 \leq \|A\|_2 \, \|u_1\|_2 = \|A\|_2$$

A normal

$\lambda_1, \cdots, \lambda_n$: eigenvalues of A

$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

$$\boxed{\|A\|_2 = |\lambda_1|}$$

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Localization Results

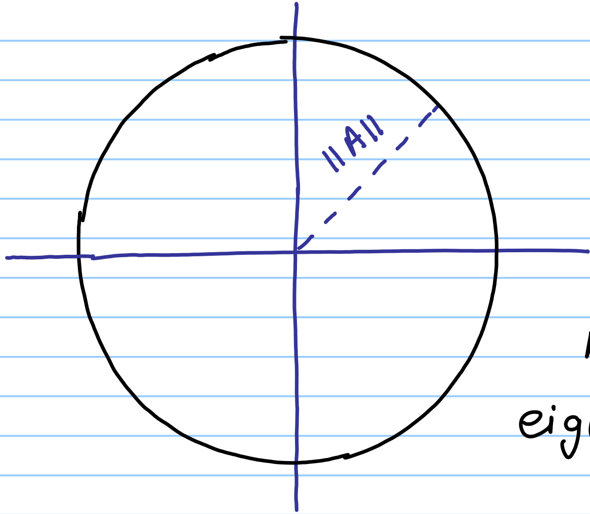
A : $n \times n$ real / complex matrix

$$Au = \lambda u, \quad \lambda \in \mathbb{C}, \quad \bar{0} \neq u \in \mathbb{C}^n$$

Induced Matrix Norm :

$$\|A\| = \max_{z \neq \bar{0}} \frac{\|Az\|}{\|z\|}$$

$$\|Au\| = \|\lambda u\| \Rightarrow |\lambda| = \frac{\|Au\|}{\|u\|} \leq \|A\|$$



The eigenvalues of A lie in $\{z \in \mathbb{C} : |z| \leq \|A\|\}$

$A^* = A \implies$ eigenvalues $\in [-\|A\|, \|A\|]$

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 6 \end{bmatrix} \qquad \|A\|_1 = \|A\|_\infty = 11$$

The eigenvalues are contained in

$[-11, 11]$

$$A = \begin{bmatrix} 4 & 1 & 0 & - & - & 0 \\ 1 & 4 & 1 & \cdots & & 0 \\ 0 & 1 & 4 & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ & & & 1 & 4 & 1 \\ 0 & - & - & - & 0 & 1 & 4 \end{bmatrix}$$

$$\|A\|_1 = \|A\|_\infty = 6$$

The eigenvalues of $A \subset [-6, 6]$

Gerschgorin Theorem

$A = [a_{ij}]$: $n \times n$ complex matrix

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\},$$

$$i = 1, \dots, n$$

Then the eigenvalues of A are contained in $\bigcup_{i=1}^{n} D_i$

Proof: $Au = \lambda u \quad , \lambda \in \mathbb{C}$

$$u = [u(1), u(2), \cdots, u(n)]^t \neq \bar{0}$$

$$(Au)(i) = \lambda u(i), \quad i = 1, \cdots, n$$

$$\sum_{j=1}^{n} a_{ij} u(j) = \lambda u(i)$$

Let $|u(k)| = \|u\|_\infty = \max_{1 \le j \le n} |u(j)|$

$$\sum_{j=1}^{n} a_{kj}\, u(j) = \lambda\, u(k)$$

$$(\lambda - a_{kk})\, u(k) = \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj}\, u(j)$$

$$\Rightarrow |\lambda - a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \underbrace{\frac{|u(j)|}{|u(k)|}}_{\leq 1}$$

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 6 \end{bmatrix}$$

$\|A\|_1 = \|A\|_\infty = 11$

evs $\in [-11, 11]$

$D_1 = \{ z \in \mathbb{C} : |z-4| \leq 3 \}$: $[1, 7]$

$D_2 = \{ z \in \mathbb{C} : |z-5| \leq 4 \}$: $[1, 9]$

$D_3 = \{ z \in \mathbb{C} : |z-6| \leq 5 \}$: $[1, 11]$

evs $\in [1, 11]$

If A is strictly row dominant:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}|, \quad i = 1, \cdots, n$$

then A is invertible.

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}$$

$$0 \notin D_i$$

A : real symmetric matrix

$$A^* = \bar{A}^t = A^t = A.$$

The eigenvalues are real.

$$Az = \lambda z, \quad \bar{0} \neq z \in \mathbb{C}^n$$

$$z = x + iy, \quad x, y \in \mathbb{R}^n$$

$$A(x+iy) = \lambda(x+iy) \Rightarrow$$

$$Ax = \lambda x, \quad Ay = \lambda y$$

$$Az = \lambda z, \quad \bar{0} \neq z \in \mathbb{C}^n$$

$$z = x + iy, \quad x, y \in \mathbb{R}^n$$

Either $x \neq \bar{0}$ or $y \neq \bar{0}$

$$Ax = \lambda x, \quad Ay = \lambda y$$

For a real symmetric matrix,
we can choose a real eigenvector

Eigenvalue Problem

To find $\lambda \in \mathbb{C}$ and $\bar{0} \neq u \in \mathbb{C}^n$ such that

$$Au = \lambda u$$

λ : known, u : solution of

$$(A - \lambda I)u = \bar{0}$$

$$Au = \lambda u$$

u : known.

Find the constant of proportionality between vectors Au and u.

$$\lambda = \frac{u^* A u}{u^* u}$$

Power Method

Assumptions : 1) The eigenvalues of A are such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

λ_1 : dominant eigenvalue of A.

2) A has n linearly independent eigenvectors : $A u_j = \lambda_j u_j$, $j = 1, \cdots, n$

If A has n distinct eigenvalues or if A is normal, then A has n linearly independent eigenvectors.

In fact, if A is normal, then A has n <u>orthonormal</u> eigenvectors.

Choose $z \neq \overline{0}$. Then

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n,$$

$$\begin{aligned} Az &= \alpha_1 A u_1 + \alpha_2 A u_2 + \cdots + \alpha_n A u_n \\ &= \lambda_1 (\alpha_1 u_1) + \lambda_2 (\alpha_2 u_2) + \cdots + \lambda_n (\alpha_n u_n) \end{aligned}$$

$$A^k z = \lambda_1^k (\alpha_1 u_1) + \lambda_2^k (\alpha_2 u_2) + \cdots + \lambda_n^k (\alpha_n u_n)$$

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

$$A^k z = \lambda_1^k \left(\alpha_1 u_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \alpha_2 u_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \alpha_n u_n \right), \quad k = 1, 2, \cdots$$

$$\frac{A^k z}{\lambda_1^k} \longrightarrow \alpha_1 u_1 \quad \text{as } k \to \infty$$

$$\frac{A^k z}{\|A^k z\|} = \frac{\lambda_1^k \left(\alpha_1 u_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \alpha_2 u_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \alpha_n u_n\right)}{|\lambda_1|^k \left\| \alpha_1 u_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \alpha_2 u_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \alpha_n u_n \right\|}$$

If $\lambda_1 > 0$, then

$$\frac{A^k z}{\|A^k z\|} \longrightarrow \frac{\alpha_1 u_1}{\|\alpha_1 u_1\|}$$

If $\lambda_1 < 0$, then even subsequence

$$\longrightarrow \frac{\alpha_1 u_1}{\|\alpha_1 u_1\|}$$

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

As z is chosen randomly,

$\alpha_1 \neq 0$.

Round-off errors $\Rightarrow$ α_1 is never 0

Let $z^{(0)} = \dfrac{z}{\|z\|}$, $z^{(k)} = \dfrac{A z^{(k-1)}}{\|A z^{(k-1)}\|}$,

$k = 1, 2, \cdots$

Claim: $z^{(k)} = \dfrac{A^k z}{\|A^k z\|}$ $\cdots (1)$

Proof: by induction

$k = 0$: OK! Assume (1) for $k = n$.

$$z^{(n+1)} = \frac{A z^{(n)}}{\|A z^{(n)}\|} = \frac{A^{n+1} z \,/\, \|A^n z\|}{\|A^{n+1} z\| \,/\, \|A^n z\|}$$

Suppose that A is invertible

and $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$.

$$A u_j = \lambda_j u_j \implies A^{-1} u_j = \frac{1}{\lambda_j} u_j,$$

$$j = 1, \ldots, n$$

$$\left|\frac{1}{\lambda_n}\right| > \left|\frac{1}{\lambda_{n-1}}\right| \geq \cdots \geq \frac{1}{|\lambda_1|}$$

Apply power method to A^{-1}.