Note Title 5/2/2011

Power Method

$A u_j = \lambda_j u_j, \quad j = 1, \cdots, n$

$\{u_1, \cdots, u_n\}$: basis for $\mathbb{C}^n$

$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$

$z \neq \bar{0}, \quad z = \alpha_1 u_1 + \cdots + \alpha_n u_n, \quad \alpha_1 \neq 0$

$\lambda_1 > 0$

$$\frac{A^k z}{\|A^k z\|} \longrightarrow \frac{\alpha_1 u_1}{\|\alpha_1 u_1\|}$$: unit $\vec{ev}$ assoc. with λ_1

Note Title 4/29/2011

Inverse Power Method

A has n linearly independent eigenvectors $A u_j = \lambda_j u_j$,

$j = 1, \cdots, n$

μ : approximation to λ_ℓ.

$|\mu - \lambda_\ell| < |\mu - \lambda_j|, \; j \neq \ell$

$$A u_j = \lambda_j u_j , \quad j = 1, \cdots, n$$

$$|\mu - \lambda_\ell| < |\mu - \lambda_j| , \quad j \neq \ell$$

$$(A - \mu I) u_j = (\lambda_j - \mu) u_j$$

$$(A - \mu I)^{-1} u_j = \frac{1}{(\lambda_j - \mu)} u_j$$

The matrix $(A - \mu I)^{-1}$ has $\dfrac{1}{\lambda_\ell - \mu}$ as the dominant eigenvalue.

Choose $z \neq \overline{0}$.

Define $z^{(0)} = \dfrac{z}{\|z\|}$,

$$z^{(k)} = \frac{(A - \mu I)^{-1} z^{(k-1)}}{\|(A - \mu I)^{-1} z^{(k-1)}\|}$$

Then $z^{(k)} \longrightarrow w$, where

$$(A - \mu I)^{-1} w = \frac{1}{\lambda_l - \mu} w$$

$$(A - \mu I)^{-1} w = \frac{1}{\lambda_l - \mu} w$$

$$\Rightarrow (A - \mu I) w = (\lambda_l - \mu) w$$

$$\Rightarrow A w = \lambda_l w$$

w : eigenvector associated with λ_l.

$$z^{(k)} = \frac{(A - \mu I)^{-1} z^{(k-1)}}{\| (A - \mu I)^{-1} z^{(k-1)} \|}$$

We need to find

$$(A - \mu I)^{-1} z^{(k-1)} = r^{(k-1)}$$

or equivalently solve

$$(A - \mu I) r^{(k-1)} = z^{(k-1)}$$

A invertible matrix.

λ, μ eigenvalues of A

$$|\lambda| \leq \|A\|, \quad \frac{1}{|\mu|} \leq \|A^{-1}\|$$

$$\Rightarrow \frac{|\lambda|}{|\mu|} \leq \|A\| \, \|A^{-1}\|$$

$$\Rightarrow \frac{|\lambda_1|}{|\lambda_n|} \leq \|A\| \, \|A^{-1}\|$$

μ : approximation to λ_ℓ.

$$\|(A-\mu I)^{-1}\| \geqslant \frac{1}{|\mu-\lambda_\ell|}$$

$A-\mu I$ is ill-conditioned.

Recall :

$$Ax = b, \quad A(x + \delta x) = b + \delta b$$

$$\frac{\|\delta x\|}{\|x\|} \leq \left(\|A\| \, \|A^{-1}\|\right) \frac{\|\delta b\|}{\|b\|}$$

sensitive to perturbation

We need to solve

$$(A - \mu I)\, r = q$$

Instead we solve

$$(A - \mu I)\, \hat{r} = \hat{q}$$

Even though $\dfrac{\| q - \hat{q} \|}{\| q \|}$ is small,

$\dfrac{\| r - \hat{r} \|}{\| r \|}$ can be big.

$$q = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

$$\hat{q} = \hat{c}_1 u_1 + \hat{c}_2 u_2 + \cdots + \hat{c}_n u_n$$

$$r = (A - \mu I)^{-1} q = \frac{c_1}{\lambda_1 - \mu} u_1 + \cdots + \frac{c_l}{\lambda_l - \mu} u_l$$

$$\cdots + \frac{c_n}{\lambda_n - \mu} u_n$$

$$\hat{r} = (A - \mu I)^{-1} \hat{q} = \frac{\hat{c}_1}{\lambda_1 - \mu} u_1 + \cdots + \frac{\hat{c}_l}{\lambda_l - \mu} u_l$$

$$\cdots + \frac{c_n}{\lambda_n - \mu} u_n$$

$$r \simeq \frac{c_\ell}{\lambda_\ell - \mu} u_\ell , \quad \hat{r} \simeq \frac{\hat{c}_\ell}{\lambda_\ell - \mu} u_\ell$$

$$\| r - \hat{r} \| \simeq \frac{| c_\ell - \hat{c}_\ell |}{| \lambda_\ell - \mu |} : \underline{\text{can}} \ \underline{\text{be}} \ \underline{\text{big}}$$

We are only interested in eigenvector, that is direction, hence it is fine

q : approximate eigenvector

What should be chosen as an approximate eigenvalue?

Rayleigh quotient: $\dfrac{q^* A q}{q^* q} = \eta$

$$\eta = \frac{q^* A q}{q^* q}$$

For $z \in \mathbb{C}$, consider

$$\| Aq - zq \|_2^2 = \| \underbrace{Aq - \eta q} + \underbrace{(\eta - z) q} \|_2^2$$

$$\langle Aq - \eta q, q \rangle = q^* A q - \eta \, q^* q = 0$$

$$\| Aq - zq \|_2^2 = \| Aq - \eta q \|_2^2 + \| (\eta - z) q \|_2^2$$

$$\eta = \frac{q^* A q}{q^* q}$$

For any $z \in \mathbb{C}$,

$$\| A q - \eta q \|_2 \le \| A q - z q \|_2$$

q : approximate eigenvector

Choose η : approximate eigenvalue

Note Title 5/1/2011

QR decomposition

A : $n \times n$ real invertible matrix

Aim : To write

$$A = QR,$$

where

Q : orthogonal, $Q^t Q = Q Q^t = I$,

R : upper triangular

$$Q = [q_1 \ q_2 \ \cdots \ q_n]$$

q_j : jth column of Q

$$Q^t Q = \begin{bmatrix} q_1^t \\ q_2^t \\ \vdots \\ q_n^t \end{bmatrix} [q_1 \ q_2 \cdots q_n]$$

$$= [q_i^t q_j]$$

$$= [\langle q_j, q_i \rangle]$$

$$= I$$

$$Q^t Q = I$$

$$\Rightarrow \langle q_j, q_i \rangle = q_i^t q_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

The columns of Q are orthonormal.

$QQ^t = I \Rightarrow$ The rows of Q are orthonormal

$A = QR$, Q orthogonal,

R : upper triangular

$$[c_1 \ c_2 \ \cdots \ c_n] = [q_1 \ q_2 \cdots q_n] \begin{bmatrix} r_{11} & \cdots & & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

$c_1 = r_{11} q_1$

$c_2 = r_{12} q_1 + r_{22} q_2$

...