Note Title 5/1/2011

QR decomposition

$A = QR$, Q orthogonal,

R: upper triangular

$$[c_1 \; c_2 \; \cdots \; c_n] = [q_1 \; q_2 \cdots q_n] \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ 0 & r_{22} \cdots & r_{2n} \\ \vdots & & \\ 0 & 0 \cdots & r_{nn} \end{bmatrix}$$

$c_1 = r_{11} q_1$

$c_2 = r_{12} q_1 + r_{22} q_2$

...

$$c_1 = r_{11} q_1$$

$$c_2 = r_{12} q_1 + r_{22} q_2$$

$$\vdots$$

$$c_j = r_{1j} q_1 + \cdots + r_{jj} q_j$$

$$\vdots$$

$$\operatorname{span}\{c_1, \cdots, c_j\} = \operatorname{span}\{q_1, \cdots, q_j\}$$

$$j = 1, \cdots, n$$

$$c_1 = r_{11} q_1 \Rightarrow \|c_1\|_2 = |r_{11}|$$

$$c_2 = r_{12} q_1 + r_{22} q_2$$

$$\langle c_2, q_1 \rangle = r_{12} \langle q_1, q_1 \rangle + r_{22} \underbrace{\langle q_2, q_1 \rangle}_{=0}$$

$$r_{12} = \langle c_2, q_1 \rangle$$

$$r_{22} q_2 = c_2 - \langle c_2, q_1 \rangle q_1$$

$$|r_{22}| = \| c_2 - \langle c_2, q_1 \rangle q_1 \|_2$$

$$C_j = r_{1j}\, q_1 + r_{2j}\, q_2 + \cdots + r_{jj}\, q_j$$

$q_1, \cdots, q_{j-1}$: orthonormal

: have been determined

$$\langle C_j, q_k \rangle = \left\langle \sum_{i=1}^{j} r_{ij}\, q_i , q_k \right\rangle$$

$$= \sum_{i=1}^{k} r_{ij} \langle q_i, q_k \rangle$$

$$= r_{kj}, \quad k = 1, \cdots, j-1$$

$$c_j = r_{1j} q_1 + \cdots + r_{j-1,j} q_{j-1} + r_{j,j} q_j$$

$$= \sum_{i=1}^{j-1} \langle c_j, q_i \rangle q_i + r_{jj} q_j$$

$$r_{jj} q_j = c_j - \sum_{i=1}^{j-1} \langle c_j, q_i \rangle q_i$$

$$\Rightarrow |r_{jj}| = \left\| c_j - \sum_{i=1}^{j-1} \langle c_j, q_i \rangle q_i \right\|_2$$

$A = [c_1, c_2, \cdots, c_n]$, $Q = [q_1, q_2, \cdots, q_n]$

$R = [r_{ij}]$: $r_{ij} = 0$ if $i > j$

Choose $r_{ii} > 0$

$r_{11} = \|c_1\|_2$, $q_1 = \dfrac{c_1}{\|c_1\|_2}$

for $j = 2, 3, \cdots, n$

$r_{ij} = \langle c_j, q_i \rangle$, $i = 1, \cdots, j-1$,

$s_j = c_j - \sum_{i=1}^{j-1} \langle c_j, q_i \rangle q_i$

$r_{jj} = \|s_j\|_2$, $q_j = \dfrac{s_j}{\|s_j\|_2}$

Gram-Schmidt

Orthonormalization

A invertible $\Rightarrow$

$A = QR$, Q : orthogonal,

R : upper triangular, $r_{ii} > 0$.

Uniqueness of QR decomposition

$$A = Q_1 R_1 = Q_2 R_2$$

$$\Rightarrow A^t = R_1^t Q_1^t = R_2^t Q_2^t$$

$$\Rightarrow A^t A = R_1^t Q_1^t Q_1 R_1 = R_1^t R_1$$

$$= R_2^t R_2$$

A invertible $\Rightarrow$ $A^t A$ positive-definite

$$A^t A = R_1^t R_1 = R_2^t R_2$$

R_1^t, R_2^t : lower triangular matrices with positive diagonal entries

Cholesky - decomposition

Hence $R_1 = R_2$

$$A = Q_1 R_1 = Q_2 R_2$$

$R_1 = R_2$: positive diagonal entries

$$Q_1 = Q_2 = R_1^{-1} A .$$

QR method

Write $A = Q_0 R_0$

Define $A_1 = R_0 Q_0$

Write $A_1 = Q_1 R_1$,

Define $A_2 = R_1 Q_1$, ...

$A_m = Q_m R_m$, Define

$A_{m+1} = R_m Q_m$

$$A_m = Q_m R_m \quad , \quad A_{m+1} = R_m Q_m$$

$$\Rightarrow \quad Q_m^t A_m Q_m = Q_m^t Q_m R_m Q_m$$

$$= R_m Q_m = A_{m+1}$$

$$A_{m+1} = Q_m^t A_m Q_m$$

$$= Q_m^t Q_{m-1}^t A_{m-1} Q_{m-1} Q_m$$

$$\cdots$$

$$= Q_m^t \cdots Q_0^t A_0 Q_0 \cdots Q_m$$

$$A_{m+1} = (Q_0 \cdots Q_m)^t A_0 (Q_0 \cdots Q_m)$$

orthogonal

$A_0 = A$ and A_{m+1} have the same eigenvalues.

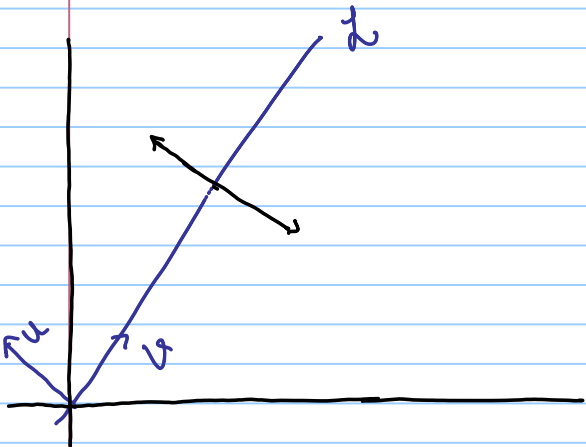
Under appropriate conditions,

$A_{m+1} \longrightarrow U$, an upper triangular matrix

Let $x, y \in \mathbb{R}^n$ be such that
$\|x\|_2 = \|y\|_2 = 1$.

Aim: To find an orthogonal matrix Q such that $Qx = y$.

Reflectors : $n = 2$



$$\|u\|_2 = \|v\|_2 = 1$$

$$\langle u, v \rangle = 0$$

To find Q such that

$$Qv = v, \quad Qu = -u$$

$$\|u\|_2 = \|v\|_2 = 1, \quad \langle u, v \rangle = v^t u = 0$$

$$P = u u^t : \quad Pu = u(u^t u) = u$$

$$Pv = u(u^t v) = u(v^t u)^t = \overline{0}$$

Let $Q = I - 2P$. Then

$$Qu = -u, \quad Qv = v.$$

$P = uu^t, \quad Q = I - 2P$

$P^2 = u \underbrace{u^t u}_{=1} u^t = uu^t = P$

projection

$P^t = uu^t = P$, Range of $P = \text{span}\{u\}$

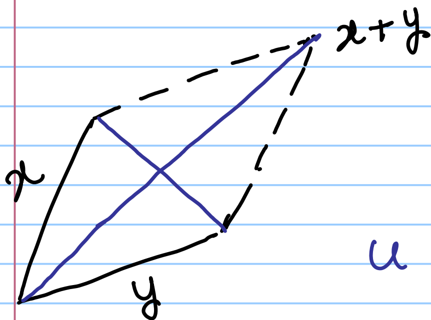
$Q^t = Q$. $\quad Q^2 = (I - 2P)(I - 2P)$

$= I - 2P - 2P + 4P^2$

$= I$ orthogonal

$$\|x\|_2 = \|y\|_2, \quad x \neq y$$

To find an orthogonal matrix Q such that $Qx = y$.



$$v = \frac{x+y}{\|x+y\|_2}$$

$$u = \frac{x-y}{\|x-y\|_2}$$

$$u = \frac{x-y}{\|x-y\|_2}, \quad v = \frac{x+y}{\|x+y\|_2}$$

$$P = u u^t, \quad Q = I - 2P$$

Then $Qv = v$, $Qu = -u$

Consider

$$Qx = Q\left(\frac{x+y}{2} + \frac{x-y}{2}\right)$$

$$= \frac{x+y}{2} - \frac{x-y}{2} = y$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\quad\;\; \| \qquad\qquad\qquad \|$$

$$\quad\;\; x \qquad\qquad\qquad y$$

$$\sigma_1 = \left(\sum_{i=1}^{n} a_{i1}^2 \right)^{1/2}$$

$$u = \frac{x - y}{\|x - y\|_2}$$

$$Q_1 = I - 2uu^t$$

$$x - y = \begin{bmatrix} a_{11} - \sigma_1 \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \quad \sigma_1^2 = \sum_{i=1}^{n} a_{i1}^2$$

$$\|x - y\|_2^2 = \sum_{i=1}^{n} a_{i1}^2 - 2\sigma_1 a_{11} + \sigma_1^2$$

$$= 2\sigma_1(\sigma_1 - a_{11}) = -2\sigma_1 u_1$$