Note Title 3/9/2012

Properties of divided difference

$f \in C[a,b]$, $x_0, x_1, \cdots, x_n \in [a,b]$

$\Rightarrow p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$ is <u>the</u> interpolating polynomial.

Define $I_n : C[a,b] \longrightarrow C[a,b]$ by

$$I_n f = p_n .$$

$I_n(\alpha f + g) = \alpha\, I_n(f) + I_n(g)$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$q_n(x_j) = g(x_j), \quad j = 0, 1, \cdots, n$$

$$\Rightarrow \Big(\alpha \, \underbrace{p_n + q_n}_{\text{polynomial of degree} \leq n}\Big)(x_j) = (\alpha f + g)(x_j)$$

$$\boxed{\begin{aligned} I_n(\alpha f + g) &= \alpha \, p_n + q_n \\ &= \alpha \, I_n(f) + I_n(g) \end{aligned}}$$

$f(x) = 1 \implies I_n(f) = f, \quad n \geq 0$

$f(x) = x \implies I_n(f) = f, \quad n \geq 1$

$f(x) = x^2 \implies I_n(f) = f, \quad n \geq 2$.

$\| I_n(f) - f \|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$

for $f(x) = 1, x, x^2$.

$f \geq 0 \not\implies I_n(f) \geq 0$

Korovkin Theorem : not applicable.

Consider $f(x) = x^2$, $x \in [0,2]$

Let $x_0 = 1$, $x_1 = 2$.

$f(x_0) = 1$, $f(x_1) = 4$

Then $p_1(x) = 4(x-1) + (2-x)$

$= 3x - 2$

interpolates f at 1 and 2.

$f(x) \geq 0$, $x \in [0,2]$, but $p_1(x) < 0$, $x \in [0, \frac{2}{3})$

Divided Difference

$f : [a, b] \longrightarrow \mathbb{R}$, $x_0, x_1, \cdots, x_n$: distinct points in $[a,b]$

Then there exists a unique polynomial p_n of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

Definition : The divided difference

$f[x_0, x_1, \cdots x_n]$ = coefficient of x^n in $p_n(x)$

f : polynomial of degree $m \le n$

$\Rightarrow \quad p_n(x) = f(x)$.

1) f : polynomial of degree $\underline{m < n}$

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$p_n(x) = f(x) = a_0 + a_1 x + \cdots + a_m x^m + 0 \cdot x^{m+1} + \cdots + 0 \cdot x^n$$

$\Rightarrow \quad f[x_0, x_1, \cdots, x_n] = 0$

2) $f[x_0, x_1, \cdots x_n]$: does not depend on the order of $x_0, x_1, \cdots, x_n$

3) Recurrence formula

$$\boxed{\begin{aligned} f[x_0] &= f(x_0) \\ f[x_0, x_1, \cdots, x_n] &= \frac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0} \end{aligned}}$$

4) $$\boxed{f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}}$$

$$\omega(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

Linearity of the divided difference

$f, g : [a, b] \longrightarrow \mathbb{R}$, $\alpha \in \mathbb{R}$

p_n interpolates f at x_j ,

q_n interpolates g at x_j , $j = 0, 1, \cdots, n$

$\Rightarrow$ $\alpha p_n + q_n$ interpolates $\alpha f + g$.

5) $$\boxed{(\alpha f + g)[x_0, x_1, \cdots, x_n] = \alpha f[x_0, x_1, \cdots, x_n] + \beta g[x_0, x_1, \cdots, x_n]}$$

Theorem: $f : [a,b] \longrightarrow \mathbb{R}$, $f, f', \cdots, f^{(n-1)}$ continuous on $[a,b]$, $f^{(n-1)}$ is differentiable on (a,b).

$x_0, x_1, \cdots, x_n$: $n+1$ distinct points in $[a,b]$.

Then $f[x_0\, x_1 \cdots x_n] = \dfrac{f^{(n)}(c)}{n!}$ for some $c \in (a,b)$.

Special case: $n = 1$: $f[x_0\, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

Mean Value Theorem $\quad = f'(c)$

Proof: Let $e(x) = f(x) - p_n(x)$,

where p_n is the interpolating polynomial.

Then $e(x_j) = f(x_j) - p_n(x_j) = 0$, $j = 0, 1, \cdots, n$.

By Rolle's theorem, $e'(y_j) = 0$, $y_j \in (x_j, x_{j+1})$

$j = 0, 1, \cdots, n-1$

e has at least $n+1$ distinct zeroes.

e' has at least n distinct zeroes.

$\vdots$

$e^{(n)}$ has at least 1 zero, $e^{(n)}(c) = 0$.

$$e(x) = f(x) - p_n(x) \quad , \quad e^{(n)}(c) = 0 \, , \quad c \in (a, b)$$

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{with}$$

$$a_n = f[x_0 \ x_1 \ \cdots \ x_n]$$

$$p_n^{(n)}(x) = n! \, a_n$$

$$e^{(n)}(c) = f^{(n)}(c) - n! \, a_n = 0 \implies$$

6) $$\boxed{f[x_0 \ x_1 \ \cdots \ x_n] = \frac{f^{(n)}(c)}{n!} \, , \quad c \in (a, b)}$$

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(c)}{n!},$$

$c \in$ smallest interval containing $x_0, x_1, \cdots, x_n$.

If $x_0 = x_1 = \cdots = x_n$, then we define

$$f[\underbrace{x_0, x_0, \cdots, x_0}_{n+1 \text{ times}}] = \frac{f^{(n)}(x_0)}{n!}$$

Continuity of the Divided Difference

Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable.

and $x_0 \in [a,b]$. For $x \in [a,b]$,

$$f[x_0, x] = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

$f[x_0, x]$ is continuous on $[a,b]$.

$$f[x_0, x_1, x] = f[x_0, x, x_1]$$

$$= \frac{f[x_1, x] - f[x_0, x]}{x_1 - x_0}$$

$\Rightarrow$ $f[x_0, x_1, x]$ is continuous.

In general,

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f[x_1, \cdots, x_n, x] - f[x_0, \cdots, x_{n-1}, x]}{x_n - x_0}.$$

By induction,

$f[x_0, x_1, \cdots, x_n, x]$ is continuous

Building of polynomials

$f : [a,b] \longrightarrow \mathbb{R}$,

p_{n-1} : polynomial of degree $\leq n-1$,

$$p_{n-1}(x_j) = f(x_j), \quad j = 0, 1, \cdots, n-1$$

p_n : polynomial of degree $\leq n$,

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n-1, n$$

Claim: $p_n(x) = p_{n-1}(x) + f[x_0, x_1, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$

Claim:

$$p_n(x) = p_{n-1}(x) + f[x_0, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$$

Proof: $p_{n-1}(x_j) = f(x_j), \quad j = 0, 1, \cdots, n-1$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$\Rightarrow \quad p_n(x_j) - p_{n-1}(x_j) = 0, \quad j = 0, 1, \cdots, n-1$$

$$\Rightarrow \quad p_n(x) - p_{n-1}(x) = \alpha (x-x_0)\cdots(x-x_{n-1})$$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$p_n(x) - p_{n-1}(x) = \alpha (x - x_0) \cdots (x - x_{n-1})$$

Coefficient of x^n in $p_n(x) - p_{n-1}(x)$

= Coefficient of x^n in $p_n(x)$

$= f[x_0, x_1, \cdots, x_n] = \alpha$

$$p_n(x) = p_{n-1}(x)$$

$$+ f[x_0, x_1, \cdots, x_n](x - x_0) \cdots (x - x_{n-1})$$

$$p_n(x) = p_{n-1}(x)$$
$$+ f[x_0, x_1, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$$

$$p_{n-1}(x) = p_{n-2}(x)$$
$$+ f[x_0, x_1, \cdots, x_{n-1}](x-x_0)\cdots(x-x_{n-2})$$

$$p_n(x) = p_{n-2}(x)$$
$$+ f[x_0, \cdots, x_{n-1}](x-x_0)\cdots(x-x_{n-2})$$
$$+ f[x_0, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$$

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

$$= \sum_{r=0}^{n} (x - x_0)\cdots(x - x_{r-1})\, f[x_0, x_1, \cdots, x_r]$$

$r = 0$ term : $f[x_0]$

Power form:

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Newton form

$$p_n(x) = \alpha_0 + \alpha_1 (x - x_0) + \alpha_2 (x - x_0)(x - x_1)$$
$$+ \cdots + \alpha_n (x - x_0) \cdots (x - x_{n-1})$$

$$\alpha_j = f[x_0, x_1, \cdots, x_j], \quad j = 0, 1, \cdots, n$$

$$x_0 = x_1 = \cdots = x_n = 0 \implies \alpha_j = \frac{f^{(j)}(0)}{j!}$$

$$p_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

Divided Difference Table

x_0	$f(x_0)$		
		$f[x_0, x_1]$	
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$
		$f[x_1, x_2]$	
x_2	$f(x_2)$		

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Divided Difference Table

x_0	$f(x_0)$		
		$f[x_0, x_1]$	
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$
		$f[x_1, x_2]$	
x_2	$f(x_2)$		

$$p_3(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= f(x_2) + f[x_2, x_1](x - x_2)$$
$$+ f[x_2, x_1, x_0](x - x_2)(x - x_1)$$

Example:

x	$f(x)$
0	1
1	4
2	15

Example:

x	$f(x)$		
0	1		
		3	
1	4		4
		11	
2	15		

$$P_2(x) = 1 + 3x + 4x(x-1)$$

$$= 15 + 11(x-2) + 4(x-2)(x-1)$$

Example:

x	$f(x)$			
0	1			
		3		
1	4		4	3
		11		
			7	
2	15			
		25		
3	40			

$$P_3(x) = P_2(x) + 3\,x(x-1)(x-2)$$

$$P_2(x) = 1 + 3\,x + 4\,x\,(x-1)$$

$$= 15 + 11\,(x-2) + 4\,(x-2)(x-1)$$

Divided Difference Table.

$$
\begin{array}{llllll}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \cdots, x_n] \\
 & & \vdots & & \cdot^{\cdot^{\cdot}} & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}
$$

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

$$= f(x_n) + f[x_{n-1}\ x_n](x - x_n) + f[x_0, \cdots, x_n](x - x_n)\cdots(x - x_1)$$

forward formula

$$p_n(x) = \sum_{r=0}^{n} (x-x_0)\cdots(x-x_{r-1})\, f[x_0, x_1, \cdots, x_r]$$

$$= \sum_{s=0}^{n} (x-x_{s+1})\cdots(x-x_n)\, f[x_s, x_{s+1}, \cdots, x_n]$$

backward formula

$s = n$ term : $f[x_n]$

Leibniz rule for the derivatives

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

$$= \sum_{r=0}^{2} \frac{2!}{r!(2-r)!} f^{(r)}(x) g^{(2-r)}(x)$$

$$\boxed{(fg)^{(n)}(x) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} f^{(r)}(x) g^{(n-r)}(x)}$$

Recall that

$$f[\underbrace{x_0, x_0, \cdots, x_0}_{n+1 \text{ times}}] = \frac{f^{(n)}(x_0)}{n!}$$

$$(fg)^{(n)}(x_0) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} f^{(r)}(x_0)\, g^{(n-r)}(x_0)$$

$$(fg)[\underbrace{x_0, \cdots, x_0}_{n+1 \text{ times}}] = \sum_{r=0}^{n} f[\underbrace{x_0, \cdots, x_0}_{r+1 \text{ times}}]\, g[\underbrace{x_0, \cdots, x_0}_{n-r+1 \text{ times}}]$$

Question

7) $(fg)[x_0, x_1, \cdots, x_n] = \sum_{r=0}^{n} f[x_0, \cdots, x_r]\, g[x_r, \cdots, x_n]$?