Note Title 3/11/2012

Leibniz Formula

Leibniz Formula for Derivatives

$$(fg)^{(n)}(x_0) = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} f^{(r)}(x_0)\, g^{(n-r)}(x_0)$$

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{n+1}] = \frac{f^{(n)}(x_0)}{n!}$$

$$(fg)[\underbrace{x_0, \cdots, x_0}_{n+1}] = \sum_{r=0}^{n} f[\underbrace{x_0, \cdots, x_0}_{r+1}]\, g[\underbrace{x_0, \cdots, x_0}_{n-r+1}]$$

$$(fg)[\underbrace{x_0, \cdots, x_0}_{n+1}] = \sum_{r=0}^{n} f[\underbrace{x_0, \cdots, x_0}_{r+1}]\, g[\underbrace{x_0, \cdots, x_0}_{n-r+1}]$$

Question :

$$(fg)[x_0, \cdots, x_n] = \sum_{r=0}^{n} f[x_0, \cdots, x_r]\, g[x_r, \cdots, x_n] \quad ?$$

Let p_n be the polynomial of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$
$$= \sum_{r=0}^{n} f[x_0, x_1, \cdots, x_r](x - x_0)\cdots(x - x_{r-1})$$

Let q_n be the polynomial of degree $\le n$ such that

$$q_n(x_j) = g(x_j), \quad j = 0, 1, \cdots, n$$

$$q_n(x) = g(x_n) + g[x_n, x_{n-1}](x - x_n)$$
$$+ \cdots + g[x_n, x_{n-1}, \cdots, x_0](x - x_n) \cdots (x - x_1)$$

$$= \sum_{s=0}^{n} g[x_n, x_{n-1}, \cdots, x_s](x - x_n) \cdots (x - x_{s+1})$$

$$p_n(x) = \sum_{r=0}^{n} f[x_0, \cdots, x_r](x-x_0)\cdots(x-x_{r-1})$$

$$q_n(x) = \sum_{s=0}^{n} g[x_n, \cdots, x_s](x-x_n)\cdots(x-x_{s+1})$$

$$p_n(x_j) = f(x_j), \quad q_n(x_j) = g(x_j),$$

$$j = 0, 1, \cdots, n$$

$$\begin{aligned}(p_n q_n)(x_j) &= p_n(x_j)\, q_n(x_j) \\ &= f(x_j)\, g(x_j), \quad j = 0, 1, \cdots, n \\ &= (fg)(x_j)\end{aligned}$$

$$(p_n q_n)(x)$$

$$= \sum_{r=0}^{n} f[x_0, \cdots, x_r](x-x_0)\cdots(x-x_{r-1})$$

$$\times \sum_{s=0}^{n} g[x_n, \cdots, x_s](x-x_n)\cdots(x-x_{s+1})$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s]$$

$$(x-x_0)\cdots(x-x_{r-1})(x-x_{s+1})\cdots(x-x_n)$$

$$(p_n q_n)(x) =$$

$$\sum_{r=0}^{n} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s]$$

$$(x - x_0) \cdots (x - x_{r-1})(x - x_{s+1}) \cdots (x - x_n)$$

$p_n q_n$: polynomial of degree $\leq 2n$

$$(p_n q_n)(x_j) = (fg)(x_j),$$

$$j = 0, 1, \cdots, n$$

$$(p_n q_n)(x) =$$

$$\sum_{r=0}^{n} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s] \, (x - x_0) \cdots (x - x_{r-1})(x - x_{s+1}) \cdots (x - x_n)$$

$$= \underbrace{\sum_{r=0}^{s} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s] \, (x - x_0) \cdots (x - x_{r-1})(x - x_{s+1}) \cdots (x - x_n)}_{T_n(x)}$$

$$+ \underbrace{\sum_{r=s+1}^{n} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s] \, (x - x_0) \cdots (x - x_{r-1})(x - x_{s+1}) \cdots (x - x_n)}_{U_n(x)}$$

$$(p_n q_n)(x) = T_n(x) + U_n(x),$$

$$U_n(x) =$$

$$\sum_{r=s+1}^{n} \sum_{s=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_s]$$

$$(x - x_0) \cdots (x - x_{r-1})(x - x_{s+1}) \cdots (x - x_n)$$

$$U_n(x_j) = 0, \quad j = 0, 1, \cdots, n$$

$$\Rightarrow (fg)(x_j) = (p_n q_n)(x_j) = T_n(x_j),$$

$$j = 0, 1, \cdots, n$$

$T_n(x) =$

$$\sum_{r=0}^{\cancel{s}} \sum_{\cancel{s}=0}^{n} f[x_0, \cdots, x_r]\, g[x_n, \cdots, x_{\cancel{s}}]$$
$$(x-x_0)\cdots(x-x_{r-1})(x-x_{\cancel{s}+1})\cdots(x-x_n)$$

T_n : polynomial of degree $\leq n$

$T_n(x_j) = (fg)(x_j)$, $j = 0, 1, \cdots, n$

$$\Rightarrow (fg)[x_0, x_1, \cdots, x_n] = \sum_{r=0}^{n} f[x_0, \cdots, x_r]\, g[x_r, \cdots, x_n]$$

Recall that

p_n : polynomial of degree $\leq n$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

p_{n+1} : polynomial of degree $\leq n+1$

$$p_{n+1}(x_j) = f(x_j), \quad j = 0, 1, \cdots, n, n+1$$

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \cdots, x_{n+1}](x - x_0) \cdots (x - x_n)$$

Error in the interpolating polynomial:

Theorem: Let $f : [a,b] \to \mathbb{R}$, $x_0, x_1, \cdots, x_n$: distinct points in $[a,b]$, p_n : polynomial of degree $\leq n$ such that $p_n(x_j) = f(x_j)$, $j = 0, 1, \cdots, n$. Then for $x \neq x_j$, $j = 0, 1, \cdots, n$,

$$\boxed{f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x](x - x_0)\cdots(x - x_n).}$$

Proof: Let $\bar{x} \neq x_j$, $j = 0, 1, \cdots, n$ and let p_{n+1} be the polynomial of degree $\leq n+1$ such that

$$p_{n+1}(x_j) = f(x_j), \; j = 0, 1, \cdots, n \quad \text{and}$$

$$p_{n+1}(\bar{x}) = f(\bar{x}).$$

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \cdots, x_n, \bar{x}](x - x_0) \cdots (x - x_n)$$

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \cdots, x_n, \bar{x}](x-x_0)\cdots(x-x_n)$$

$\Rightarrow$

$$f(\bar{x}) = p_{n+1}(\bar{x}) = p_n(\bar{x}) +$$

$$+ f[x_0, x_1, \cdots, x_n, \bar{x}](\bar{x}-x_0)\cdots(\bar{x}-x_n)$$

$\Rightarrow$ For $x \neq x_j$,

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x](x-x_0)\cdots(x-x_n)$$

For $x = x_j$, $f(x_j) - p_n(x_j) = 0$, $j = 0, 1, \cdots, n$.

Error in the interpolating polynomial

$$f(x) - p_n(x) = f[x_0, \cdots, x_n, x] \underbrace{(x-x_0)\cdots(x-x_n)}_{\omega(x)}$$

$$\| f - p_n \|_\infty = \max_{x \in [a,b]} | f[x_0, \cdots, x_n, x] |$$

$$\| \omega \|_\infty$$

Problem: To choose $x_0, \cdots, x_n$ such that $\| \omega \|_\infty$ is minimized

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$
$$= x^{n+1} - (a_0 + a_1 x + \cdots + a_n x^n)$$
$$= x^{n+1} - q_n(x)$$

$$\min_{\substack{\{x_0, \cdots x_n\} \\ \subset [a,b]}} \|\omega\|_\infty = \min_{q_n \in \mathbb{P}_n} \|x^{n+1} - q_n(x)\|_\infty$$

$\mathbb{P}_n$: polynomials of degree $\leq n$

Chebyshev Polynomials

$T_n(x) = \cos(n\theta)$ when

$x = \cos(\theta)$, $0 \leq \theta \leq \pi$, $x \in [-1, 1]$

$T_n(x) = \cos(n \cos^{-1} x)$, $x \in [-1, 1]$

$T_{n+1}(x) + T_{n-1}(x) = \cos((n+1)\theta) + \cos((n-1)\theta)$

$= 2\cos(n\theta)\cos(\theta)$

$T_{n+1}(x) = 2x\,T_n(x) + T_{n-1}(x)$

$T_n(x) = \cos(n\theta)$ when $x = \cos\theta$

$T_0(x) = 1, \quad T_1(x) = x$

$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

T_{n+1} : polynomial of degree $n+1$ with the leading coefficient 2^n.

Proof is by induction.

True for $n = 0, 1$. Then use recurrence formula

Roots of the Chebyshev Polynomial

$T_n(x) = \cos(n\theta)$ when $x = \cos\theta$

$\cos(n\theta) = 0$ if $n\theta = \frac{\pi}{2} + m\pi$,

$m = 0, \pm 1, \pm 2, \cdots$

Distinct roots : $\cos(\theta)$ with

$$n\theta = \frac{\pi}{2}, \frac{\pi}{2} + \pi, \cdots, \frac{\pi}{2} + (n-1)\pi,$$

$$\theta = \frac{1}{n}\left(\frac{\pi}{2} + m\pi\right), \quad m = 0, 1, \cdots, (n-1)$$

Roots of $T_n(x)$: $\cos\theta$ with

$$\theta = \frac{1}{n}\left(\frac{\pi}{2} + m\pi\right), \quad m = 0, 1, \dots, n-1$$

$$= \frac{(2m+1)\pi}{2n}$$

Divide upper part of the unit circle into n equal parts : $0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \frac{n\pi}{n}$

Take Midpoints : $\frac{\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{(2n-1)\pi}{2n}$

Project onto $[-1, 1]$