Note Title 3/14/2012

Cubic Hermite Interpolation

Newton Form

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

$$= a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

Number of Computations

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

2 Subtractions + 1 division

Divided Difference Table

x_0	$f(x_0)$			
		$f[x_0, x_1]$		
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		
x_2	$f(x_2)$		$\vdots$	$f[x_0, \cdots, x_n]$
$\vdots$	$\vdots$	$\vdots$		
			$f[x_{n-2}, x_{n-1}, x_n]$	
		$f[x_{n-1}, x_n]$		
x_n	$f(x_n)$			

$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$ divided differences

$\frac{n(n+1)}{2}$ divided differences

2 subtractions + 1 division

for each divided difference

Total Cost for the divided

difference table:

$n(n+1)$ subtractions +

$\frac{n(n+1)}{2}$ divisions

Horner's Scheme

$$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

3 multiplications + 2 additions

$$p_2(x) = \left[a_2(x - x_1) + a_1\right](x - x_0) + a_0$$

2 multiplications + 2 additions

$$p_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

$1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ multiplications

$+\ n$ additions

$b_n = a_n$. For $j = n-1, \dots, 0$

$$b_j = a_j + b_{j+1}(x - x_j)$$

Then $b_0 = p_n(x)$ $\quad$ n multiplications

$+\ n$ additions

Cubic Hermite Interpolation

$f : [a, b] \longrightarrow \mathbb{R}$ is differentiable

Aim: To find a polynomial p_3 of degree ≤ 3 such that

$p_3(a) = f(a)$, $p_3(b) = f(b)$,

$p_3'(a) = f'(a)$, $p_3'(b) = f'(b)$.

Recall that

$$p_3(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$x_0 = x_1 = a \quad , \quad x_2 = x_3 = b$

$f[a, a] = f'(a) \,, \quad f[a, a, b] = \dfrac{f[a, b] - f'(a)}{b - a}$

$$
\begin{array}{llllll}
a & f(a) & & & \\
  &   & f'(a) & & \\
a & f(a) & & f[a,a,b] & \\
  &   & f[a,b] & & f[a,a,b,b] \\
b & f(b) & & f[a,b,b] & \\
  &   & f'(b) & & \\
b & f(b) & & &
\end{array}
$$

$$f[a,a,b] = \frac{f[a,b] - f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f[a,b,b] - f[a,a,b]}{b-a}$$

$$p_3(x) = f(a) + f[a,a](x-a)$$
$$+ f[a,a,b](x-a)^2$$
$$+ f[a,a,b,b](x-a)^2(x-b)$$

$$p_3(a) = f(a)$$

$$p_3'(x) = f[a,a] + 2f[a,a,b](x-a)$$
$$+ f[a,a,b,b]\left\{2(x-a)(x-b) + (x-a)^2\right\}$$

$$p_3'(a) = f'(a)$$

$$p_3(x) = f(a) + f[a,a](x-a)$$
$$+ f[a,a,b](x-a)^2$$
$$+ f[a,a,b,b](x-a)^2(x-b)$$

$$p_3(b) = f(a) + f'(a)(b-a) +$$
$$+ \frac{f[a,b] - f'(a)}{b-a}(b-a)^2$$
$$= f(a) + f(b) - f(a) = f(b)$$

$$p_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

$$p_3'(x) = f[a,a] + 2f[a,a,b](x-a) + f[a,a,b,b]\left\{2(x-a)(x-b) + (x-a)^2\right\}$$

$$p_3'(b) = f'(a) + 2f[a,a,b](b-a) + \left\{f[a,b,b] - f[a,a,b]\right\}(b-a)$$

$$
\begin{aligned}
p_3'(b) &= f'(a) + 2f[a,a,b](b-a) \\
&\quad + \{ f[a,b,b] - f[a,a,b] \}(b-a) \\
&= f'(a) + \{ f[a,b,b] + f[a,a,b] \}(b-a) \\
&= f'(a) + \{ f'(b) - f[a,b] + f[a,b] - f'(a) \} \\
&= f'(b)
\end{aligned}
$$

Recall that

f has a simple zero at c
if $f(c) = 0$, $f'(c) \neq 0$.

f has a zero of multiplicity m
at c if $f(c) = f'(c) = \cdots = f^{(m-1)}(c) = 0$,
$f^{(m)}(c) \neq 0$

A polynomial of degree n has exactly n zeroes, counted according to their multiplicities.

A non-zero polynomial of degree $\leq n$ has at most n distinct zeroes.

If a polynomial of degree $\leq n$ has <u>more than</u> n zeroes, then it is a zero polynomial.

Uniqueness of the cubic Hermite Polynomial

Let $p_3(x)$ and $q_3(x)$ be polynomials of degree ≤ 3 such that

$$p_3(a) = f(a) = q_3(a), \quad p_3(b) = f(b) = q_3(b)$$

$$p_3'(a) = f'(a) = q_3'(a), \quad p_3'(b) = f'(b) = q_3'(b)$$

$p_3 - q_3$: polynomial of degree ≤ 3 such that

$$(p_3 - q_3)(a) = (p_3 - q_3)'(a) = 0$$

$$(p_3 - q_3)(b) = (p_3 - q_3)'(b) = 0$$

Thus $p_3 - q_3$ has 4 zeroes, double zero at a, double zero at b

$$\Rightarrow (p_3 - q_3)(x) \equiv 0 \Rightarrow p_3(x) = q_3(x)$$

Error :

$$f(x) - p_3(x)$$
$$= f[x_0, x_1, x_2, x_3, x]\,\omega(x),$$

where $\omega(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$

$x_0 = x_1 = a, \quad x_2 = x_3 = b$

$$f(x) - p_3(x)$$
$$= f[a, a, b, b, x](x-a)^2(x-b)^2$$

$$f(x) - p_3(x)$$

$$= f[a, a, b, b, x](x-a)^2(x-b)^2$$

$$f[a, a, b, b, x] = \frac{f^{(4)}(c_x)}{4!},$$

$$c_x \in [a, b]$$

$$\| f - p_3 \|_\infty \leq \frac{\| f^{(4)} \|_\infty}{4!} \max_{x \in [a,b]} \left| (x-a)^2(x-b)^2 \right|$$

$$\| f - p_3 \|_\infty$$

$$\leq \frac{\| f^{(4)} \|_\infty}{4!} \max_{x \in [a,b]} \left| (x-a)^2 (x-b)^2 \right|$$

Let $g(x) = (x-a)^2 (x-b)^2$

$$g'(x) = 2(x-a)(x-b)^2 + 2(x-a)^2(x-b)$$

$$= 2(x-a)(x-b)\left[x - b + x - a\right]$$

$$= 4(x-a)(x-b)\left(x - \frac{a+b}{2}\right)$$

$$g(x) = (x-a)^2(x-b)^2$$

$$g'(x) = 4(x-a)(x-b)\left(x - \frac{a+b}{2}\right)$$

$$g'\left(\frac{a+b}{2}\right) = 0$$

End points : $g(a) = g(b) = 0$

Critical point : $g\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^4$

$$\max_{x \in [a,b]} |g(x)| = \left(\frac{b-a}{2}\right)^4$$

Error in the Cubic Hermite Polynomial

$$\|f - p_3\|_\infty$$

$$\leq \frac{\|f^{(4)}\|_\infty}{4!} \max_{x \in [a,b]} \left|(x-a)^2 (x-b)^2\right|$$

$$\leq \frac{\|f^{(4)}\|_\infty}{4!} \left(\frac{b-a}{2}\right)^4$$

$$\begin{array}{lllll}
a & f(a) & & & \\
 & & f'(a) & & \\
a & f(a) & & f[a,a,b] & \\
 & & f[a,b] & & f[a,a,b,b] \\
b & f(b) & & f[a,b,b] & \\
 & & f'(b) & & \\
b & f(b) & & &
\end{array}$$

$$f[a,a,b] = \frac{f[a,b] - f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f[a,b,b] - f[a,a,b]}{b-a}$$

$$
\begin{array}{llllll}
a & f(a) & & & \\
 & & f'(a) & & \\
a & f(a) & & f[a,a,b] & \\
 & & f[a,b] & & f[a,a,b,b] \\
b & f(b) & & f[a,b,b] & \\
 & & f'(b) & & \\
b & f(b) & & &
\end{array}
$$

$$
\begin{aligned}
P_3(x) = {} & f(a) + f'(a)(x-a) \\
& + f[a,a,b](x-a)^2 \\
& + f[a,a,b,b](x-a)^2(x-b)
\end{aligned}
$$

Example

$$f(x) = x^4 + x^3 + x^2 + x + 1$$

$$f'(x) = 4x^3 + 3x^2 + 2x + 1$$

0	1			
		1		
0	1		3	
		4		3
1	5		6	
		10		
1	5			

$$p_3(x) = 1 + x + 3x^2 + 3x^2(x-1)$$

Example

$$f(x) = x^4 + x^3 + x^2 + x + 1$$

0	1				
		1			
0	1		3		
		4		3	
1	5		6		1
		10		5	
1	5		16		
		26			
2	31				

$$p_3(x) = 1 + x + 3x^2 + 3x^2(x-1)$$
$$= 1 + x + 3x^3$$

$$p_4(x) = p_3(x) + x^2(x-1)^2$$
$$= 1 + x + 3x^3 + x^4 - 2x^3 + x^2$$
$$= f(x)$$

Convergence of the interpolating polynomial

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x]\, \omega(x),$$

where $\omega(x) = (x - x_0) \cdots (x - x_n)$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(c_x)}{(n+1)!}$$

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}\, \omega(x)$$

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}\,\omega(x)$$

$$\| f - p_n \|_\infty \leq \frac{\| f^{(n+1)} \|_\infty}{(n+1)!} \| \omega \|_\infty$$

$$|\omega(x)| = |(x - x_0) \cdots (x - x_n)|$$

$$\leq (b-a)^{n+1}$$

$\| f^{(n+1)} \|_\infty \leq M$ for all n $\Rightarrow \| f - p_n \|_\infty \longrightarrow 0$

$f(x) = e^x, \ \sin x$

Rungé Example

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1, 1]$$

Interpolation Points : Equidistant

$-1 \quad 1 \qquad p_1$

$-1 \quad 0 \quad 1 \qquad p_2$

$\vdots$

$-1 \quad -1+\frac{2}{n} \quad -1+\frac{4}{n} \quad \dots \quad 1 \qquad p_n$

$\|f - p_n\|_\infty \longrightarrow \infty$ as $n \to \infty$

Choose Interpolation Points

$x_{0,0}$

$x_{0,1} \quad x_{1,1}$

$\vdots$

$x_{0,n} \quad x_{1,n} \quad \cdots \quad x_{n,n}$

There exists a continuous function f for which $\|f - p_n\|_\infty \not\to 0$ as $n \to \infty$

Fix a continuous function

There exists

$x_{0,0}$

$x_{0,1} \quad x_{1,1}$ $\qquad\qquad$ $\|f - p_n\|_\infty \longrightarrow 0$

$\vdots$ $\qquad\qquad$ as $n \to \infty$.

$x_{0,n} \quad x_{1,n} \quad \cdots \quad x_{n,n}$

$\vdots$

Linear Polynomial Interpolation

$f : [a, b] \longrightarrow \mathbb{R}$

$x_0 = a, \quad x_1 = b$

$p_1(x) = f(a) + f[a, b](x - a)$

$f(x) - p_1(x) = f[a, b, x](x-a)(x-b)$

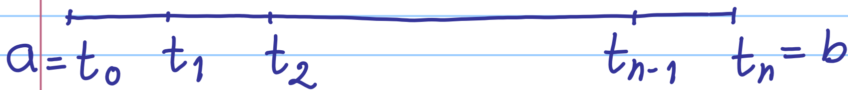
$= \left[f''(c_x)/_2 \right] (x-a)(x-b)$

$$\| f - p_1 \|_\infty \leq \frac{\| f'' \|_\infty}{2} \left(\frac{b-a}{2} \right)^2$$

Piecewise Linear Interpolation

$f : [a,b] \longrightarrow \mathbb{R}, \quad h = \dfrac{b-a}{n},$

$t_i = a + ih, \quad i = 0, 1, \cdots, n$



$g_n \in C[a,b], \quad g_n \big|_{[t_i, t_{i+1}]}$ poly. of degree ≤ 1

$g_n(t_i) = f(t_i), \quad i = 0, 1, \cdots, n$

$$\|f - p_1\|_\infty \leq \frac{\|f''\|_\infty}{2}\left(\frac{b-a}{2}\right)^2$$

$$\max_{x\in[t_i,t_{i+1}]} |f(x) - g_n(x)| \leq \frac{\|f''\|_\infty}{2}\left(\frac{h}{2}\right)^2,$$

$$\|f''\|_\infty = \max_{x\in[a,b]} |f''(x)|$$

$$\|f - g_n\|_\infty \leq \frac{\|f''\|_\infty h^2}{8} \longrightarrow 0 \text{ as } n \to \infty$$