Note Title 3/9/2011

Piecewise Polynomial Approximation

Note Title 3/9/2011

Notation : $k \geq 1$,

$C^k[a,b] = \{ f : [a,b] \to \mathbb{R} : f$ is k-times continuously differentiable $\}$.

$f \in C[a,b]$: $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$

Throughout this lecture we consider the following uniform partition of $[a,b]$:

$$a = t_0 < t_1 < \cdots < t_n = b, \quad h = t_{i+1} - t_i = \frac{b-a}{n}$$

Error in $p_n(x)$

$f : [a,b] \longrightarrow \mathbb{R}$, $x_0, x_1, \cdots, x_n$: distinct points

p_n : polynomial of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$$\boxed{f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x](x - x_0)\cdots(x - x_n)}$$

$f \in C^{n+1}[a,b] \Rightarrow$

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)\cdots(x - x_n)}$$

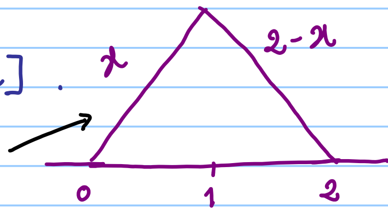
Joining of two linear polynomials

Consider two polynomials of degree 1 given by

$$p_1(x) = a_1 x + b_1 , \quad x \in [0,1]$$

and

$$q_1(x) = c_1 x + d_1 , \quad x \in [1,2] .$$



$$p_1(1) = q_1(1) \Rightarrow a_1 + b_1 = c_1 + d_1$$

$$p_1'(1) = q_1'(1) = a_1 = c_1$$: <u>Same polynomial</u>

$$c_1 = a_1 , \quad d_1 = b_1$$

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Quadratic polynomials:

$p_2(x) = a_0 + a_1 x + a_2 x^2, \quad q_2(x) = b_0 + b_1 x + b_2 x^2$

$p_2 : [0,1] \to \mathbb{R}, \quad q_2 : [1,2] \to \mathbb{R}$

Continuity: $p_2(1) = q_2(1) \Rightarrow a_0 + a_1 + a_2 = b_0 + b_1 + b_2$

Continuity of the derivative:

$p_2'(1) = q_2'(1) \Rightarrow a_1 + 2a_2 = b_1 + 2b_2$

Continuity of the second derivative:

$p_2''(1) = q_2''(1) \Rightarrow 2a_2 = 2b_2$ Same Polynomial

Piecewise polynomial space

$a = t_0 < t_1 < \cdots < t_n = b$.

$X_n = \{ g : [a,b] \to \mathbb{R} : g|_{[t_i, t_{i+1}]}$ poly. of degree $\leq k \}$.

dimension of $X_n = n(k+1)$

$Y_n = \{ g \in C[a,b] : g|_{[t_i, t_{i+1}]}$ polynomial of degree $\leq k \}$

dimension of $Y_n = n(k+1) - (n-1) = nk + 1$

$Z_n = \{ g \in C^1[a,b] : g|_{[t_i, t_{i+1}]}$ polynomial of degree $\leq k \}$

dimension of $Z_n = n(k+1) - 2(n-1) = nk - n + 2$

Note Title 7/30/2009

Linear Polynomial Interpolation

$f : [a, b] \longrightarrow \mathbb{R}$, $p_1(x) = a_0 + a_1 x$ such that

$$p_1(a) = f(a), \quad p_1(b) = f(b).$$

$$p_1(x) = f(a) + f[a, b](x - a)$$

Let f be twice differentiable. Then

$$f(x) - p_1(x) = f[a, b, x](x-a)(x-b) = \frac{f''(c_x)}{2}(x-a)(x-b).$$

$$\Rightarrow \max_{x \in [a,b]} |f(x) - p_1(x)| = \|f - p_1\|_\infty \leq \frac{\|f''\|_\infty}{2}\left(\frac{b-a}{2}\right)^2$$

Piecewise linear Interpolation

$a = t_0 < t_1 < \cdots < t_n = b$: uniform partition

$$t_{i+1} - t_i = h = \frac{b-a}{n}, \quad i = 0, 1, \cdots, n-1$$

Let

$$X_n = \{ g \in C[a,b] : g|_{[t_i, t_{i+1}]} \text{ poly. of degree } \leq 1 \}$$

Then $\dim X_n = 2n - (n-1) = n+1$.

Let $f : [a,b] \to \mathbb{R}$. There exists a unique $g_n \in X_n$ such that $g_n(t_i) = f(t_i)$, $i = 0, 1, \cdots, n-1$

Let $f \in C^2[a,b]$. $h = \dfrac{b-a}{n} = t_{i+1} - t_i$

For $x \in [t_i, t_{i+1}]$, $i = 0, 1, \cdots, n-1$.

$$g_n(x) = f(t_i) + f[t_i, t_{i+1}](x - t_i)$$

Hence

$$f(x) - g_n(x) = \frac{f''(c_x)}{2}(x - t_i)(x - t_{i+1}),$$

$c_x \in [t_i, t_{i+1}]$

$$\max_{x \in [t_i, t_{i+1}]} |f(x) - g_n(x)| \le \frac{\|f''\|_\infty h^2}{8}$$

$$\Rightarrow \|f - g_n\|_\infty \le \frac{\|f''\|_\infty}{8} h^2 \longrightarrow 0 \text{ as } n \to \infty.$$

$f(x) = \sqrt{x}, \quad x \in [1,2]$

Note Title 1/5/2011

Divide $[1,2]$ in n equal parts : $h = \frac{1}{n}$.

Error in the piecewise linear approximation :

$$\|f - g_n\|_\infty \le \frac{\|f''\|_\infty}{8} h^2$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4x^{3/2}} \implies \|f''\|_\infty = \frac{1}{4}.$$

In order to achieve $\|f - g_n\|_\infty < 10^{-6}$,

it suffices to choose $\frac{1}{32 n^2} < 10^{-6}$, that is,

$n^2 > \frac{10^6}{32}$: $\underline{n = 200}$ $\underline{\text{will}}$ $\underline{\text{work}}$.

Quadratic Polynomials.

Note Title 1/11/2011

$f : [a, b] \longrightarrow \mathbb{R}$, p_2 : polynomial of degree ≤ 2

such that

$$p_2(a) = f(a), \quad p_2\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right), \quad p_2(b) = f(b).$$

$$p_2(x) = f(a) + f\left[a \ \ \frac{a+b}{2}\right](x-a)$$

$$+ f\left[a \ \ \frac{a+b}{2} \ \ b\right](x-a)\left(x - \frac{a+b}{2}\right)$$

$$f(x) - p_2(x) = f\left[a \ \ \frac{a+b}{2} \ \ b \ \ x\right](x-a)\left(x - \frac{a+b}{2}\right)(x-b)$$

$$= \frac{f'''(c_x)}{3!} \overbrace{(x-a)\left(x - \frac{a+b}{2}\right)(x-b)}^{\omega(x)}$$

$$\|w\|_\infty = \max_{x \in [a,b]} \left| (x-a)\left(x - \frac{a+b}{2}\right)(x-b) \right|$$

$$y = x - \frac{a+b}{2}, \quad k = \frac{b-a}{2}$$

$$= \max_{y \in [-k,k]} \left| (y+k)\, y\, (y-k) \right|$$

$$= \max_{y \in [-k,k]} \left| y\,(y^2 - k^2) \right| = \frac{2k^3}{3\sqrt{3}}$$

Critical point : $3y^2 - k^2 = 0 \Rightarrow y = \pm \frac{k}{\sqrt{3}}$

$$\|f - p_2\|_\infty \leq \frac{\|f'''\|_\infty}{9\sqrt{3}} \left(\frac{b-a}{2}\right)^3$$

Piecewise Quadratic Polynomial Space

Let

$$X_n = \{ f \in C[a,b] : f|_{[t_i, t_{i+1}]} \text{ polynomial of degree} \leq 2 \}$$

Then

the dimension of $X_n = 3n - (n-1) = 2n+1$

Let $s_i = \dfrac{t_i + t_{i+1}}{2}$, $i = 0, 1, \cdots, n-1$.

There exists a unique $g_n \in X_n$ such that

$$g_n(t_i) = f(t_i), \quad i = 0, 1, \cdots, n, \quad g_n(s_i) = f(s_i)$$

Recall that $\max_{x\in[a,b]} |f(x) - p_2(x)| \le \frac{\|f'''\|_\infty}{9\sqrt{3}}\left(\frac{b-a}{2}\right)^3$.

$$\|f'''\|_\infty = \max_{x\in[a,b]} |f'''(x)| .$$

Since $t_{i+1} - t_i = h$, $i = 0, 1, \cdots, n-1$,

$$\max_{x\in[t_i, t_{i+1}]} |f(x) - g_n(x)| \le \frac{\|f'''\|_\infty}{9\sqrt{3}}\left(\frac{h}{2}\right)^3$$ and hence

$$\max_{x\in[a,b]} |f(x) - g_n(x)| \le \frac{\|f'''\|_\infty}{9\sqrt{3}}\left(\frac{h}{2}\right)^3 \longrightarrow 0 \text{ as } n \to \infty.$$

$f(x) = \sqrt{x}, \quad x \in [1, 2]$. $\quad \| f - g_n \|_\infty \le \dfrac{\| f''' \|_\infty}{72\sqrt{3}} h^3$

$f'(x) = \dfrac{1}{2\sqrt{x}}, \quad f''(x) = -\dfrac{1}{4x^{3/2}}, \quad f'''(x) = \dfrac{3}{8x^{5/2}}$.

$\| f''' \|_\infty = \dfrac{3}{8}, \quad \| f - g_n \|_\infty \le \dfrac{1}{192\sqrt{3}} \left(\dfrac{1}{n}\right)^3$

$\| f - g_n \|_\infty < 10^{-6}$ if $n^3 > \dfrac{10^6}{192\sqrt{3}}$

$n = 20$ will work.