Note Title 3/10/2011

# Cubic Hermite Interpolation

$f : [a, b] \longrightarrow \mathbb{R}$ differentiable.

$$\left.\begin{aligned} f(x) = f(a) + f'(a)(x-a) + f[a,a,b](x-a)^2 \\ + f[a,a,b,b](x-a)^2(x-b) \end{aligned}\right\} p_3(x)$$

$$+ \left. f[a,a,b,b,x](x-a)^2(x-b)^2 \right\} \text{error}$$

$p_3(a) = f(a)$, $p_3(b) = f(b)$, $p_3'(a) = f'(a)$, $p_3'(b) = f'(b)$

$f \in C^4[a,b] \Longrightarrow$

$$\| f - p_3 \|_\infty \le \frac{\| f^{(4)} \|_\infty}{4!} \left( \frac{b-a}{2} \right)^4$$

Note Title 3/8/2011

# Piecewise Cubic Hermite Interpolation

$$X_n = \{ g \in C^1[a,b] : g|_{[t_i, t_{i+1}]} \text{ poly. of degree} \le 3 \}$$

dimension of $X_n = 4n - 2(n-1) = 2n+2$

There exists a unique $g_n \in X_n$ such that

$$g_n(t_i) = f(t_i), \quad g_n'(t_i) = f'(t_i),$$

$$i = 0, 1, \cdots, n$$

For $x \in [t_i, t_{i+1}]$,

$$g_n(x) = f(t_i) + f'(t_i)(x - t_i) + f[t_i, t_i, t_{i+1}](x - t_i)^2$$
$$+ f[t_i, t_i, t_{i+1}, t_{i+1}](x - t_i)^2 (x - t_{i+1}).$$

Hence

$$\max_{x \in [t_i, t_{i+1}]} |f(x) - g_n(x)| \le \frac{\max_{x \in [t_i, t_{i+1}]} |f^{(4)}(x)|}{4!} \left(\frac{t_{i+1} - t_i}{2}\right)^4$$

$$\le \frac{\|f^{(4)}\|_\infty}{4!} \left(\frac{h}{2}\right)^4, \quad i = 0, 1, \cdots, n-1.$$

$$\max_{x \in [t_i, t_{i+1}]} |f(x) - g_n(x)| \leq \frac{\|f^{(4)}\|_\infty}{4!} \left(\frac{h}{2}\right)^4$$

$$\|f - g_n\|_\infty = \max_{x \in [a,b]} |f(x) - g_n(x)|$$

$$= \max_{1 \leq i \leq n-1} \max_{x \in [t_i, t_{i+1}]} |f(x) - g_n(x)|$$

$$\leq \frac{\|f^{(4)}\|_\infty}{4!} \left(\frac{h}{2}\right)^4$$

**Piecewise Linear (Continuous)**

$$\max_{x\in[a,b]} |f(x)-g_n(x)| \le \frac{\|f''\|_\infty h^2}{8} \to 0 \text{ as } n\to\infty.$$

**Piecewise Quadratic (Continuous)**

$$\max_{x\in[a,b]} |f(x)-g_n(x)| \le \frac{\|f'''\|_\infty}{72\sqrt{3}} h^3 \to 0 \text{ as } n\to\infty.$$

**Piecewise Cubic (Continuously Differentiable)**

$$\max_{x\in[a,b]} |f(x)-g_n(x)| \le \frac{\|f^{(4)}\|_\infty h^4}{384} \to 0 \text{ as } n\to\infty.$$

$f'(t_i)$ : $i=0,1,\dots,n$ may not be known.

# Cubic Spline Interpolation

$a = t_0 < t_1 < \cdots < t_n = b$ : uniform partition

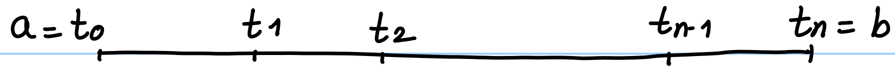
$X_n = \{ g \in C^2[a,b] : g|_{[t_i, t_{i+1}]}$ poly. of degree $\leq 3 \}$

$\dim X_n = 4n - 3(n-1) = n+3$

Aim: To find $g_n \in X_n$ such that

$$g_n(t_i) = f(t_i), \quad i = 0, 1, \cdots, n.$$

In order to have uniqueness, we need to add two extra conditions.




$g_n \in C^2[a,b]$ , $g_n$ : piecewise cubic

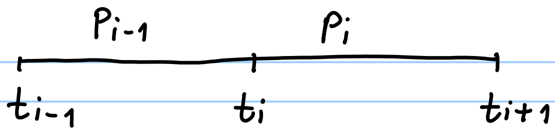
$$g_n(t_i) = f(t_i), \quad i = 0, 1, \cdots, n \,.$$

$g_n$ is completely determined if we know

$$g_n'(t_i), \quad i = 0, 1, \cdots, n$$

For $x \in [t_i, t_{i+1}]$ :

$$g_n(x) = g_n(t_i) + g_n'(t_i)(x-t_i) + g_n[t_i, t_i, t_{i+1}](x-t_i)^2$$
$$+ g_n[t_i, t_i, t_{i+1}, t_{i+1}](x-t_i)^2(x-t_{i+1})$$




Let $g_n \big|_{[t_i,\, t_{i+1}]} = P_i \,, \quad g_n \big|_{[t_{i-1},\, t_i]} = P_{i-1} \,.$

$g_n \in C^2[a,b]$ if $P_i''(t_i+) = P_{i-1}''(t_i-)$ .

$$P_i(x) = g_n(t_i) + g_n'(t_i)(x-t_i) + g_n[t_i, t_i, t_{i+1}](x-t_i)^2$$
$$+ g_n[t_i, t_i, t_{i+1}, t_{i+1}](x-t_i)^2(x-t_{i+1})$$

$$P_i'(x) = g_n'(t_i) + g_n[t_i, t_i, t_{i+1}]\, 2(x-t_i)$$
$$+ g_n[t_i, t_i, t_{i+1}, t_{i+1}]\left\{ 2(x-t_i)(x-t_{i+1}) + (x-t_i)^2 \right\}$$

$$P_i'(x) = g_n'(t_i) + g_n[t_i, t_i, t_{i+1}]\, 2(x-t_i)$$
$$+ g_n[t_i, t_i, t_{i+1}, t_{i+1}]\left\{2(x-t_i)(x-t_{i+1}) + (x-t_i)^2\right\}$$

$$\Rightarrow P_i''(x) = 2 g_n[t_i, t_i, t_{i+1}] +$$
$$g_n[t_i, t_i, t_{i+1}, t_{i+1}]\left\{4(x-t_i) + 2(x-t_{i+1})\right\},$$

$$P_i''(t_i+) = 2 g_n[t_i, t_i, t_{i+1}] - 2h\, g_n[t_i, t_i, t_{i+1}, t_{i+1}]$$

$$P_i''(t_{i+1}-) = 2 g_n[t_i, t_i, t_{i+1}] + 4h\, g_n[t_i, t_i, t_{i+1}, t_{i+1}]$$

$$P_{i-1}''(t_i-) = 2 g_n[t_{i-1}, t_{i-1}, t_i] + 4h\, g_n[t_{i-1}, t_{i-1}, t_i, t_i]$$

Equate $P_i''(t_i+) = P_{i-1}''(t_i-)$

$$p_i''(t_i+) = 2\, g_n[t_i, t_i, t_{i+1}] - 2h\, g_n[t_i, t_i, t_{i+1}, t_{i+1}]$$

$$= 4\, g_n[t_i, t_i, t_{i+1}] - 2\, g_n[t_i, t_{i+1}, t_{i+1}]$$

$$= \frac{4}{h}\left\{ g_n[t_i, t_{i+1}] - g_n'(t_i) \right\} - \frac{2}{h}\left\{ g_n'(t_{i+1}) - g_n[t_i, t_{i+1}] \right\}$$

$$= \frac{6\, g_n[t_i, t_{i+1}] - 4\, g_n'(t_i) - 2\, g_n'(t_{i+1})}{h}$$

$$p_{i-1}''(t_i-) = 2\, g_n[t_{i-1}, t_{i-1}, t_i] + 4h\, g_n[t_{i-1}, t_{i-1}, t_i, t_i]$$

$$= -2\, g_n[t_{i-1}, t_{i-1}, t_i] + 4\, g_n[t_{i-1}, t_i, t_i]$$

$$= \frac{-6\, g_n[t_{i-1}, t_i] + 4\, g_n'(t_i) + 2\, g_n'(t_{i-1})}{h}$$

$$P_i''(t_i+) = \frac{6\, g_n[t_i, t_{i+1}] - 4\, g_n'(t_i) - 2\, g_n'(t_{i+1})}{h}$$

$$\shortparallel$$

$$P_{i-1}''(t_i-) = \frac{-6\, g_n[t_{i-1}, t_i] + 4\, g_n'(t_i) + 2\, g_n'(t_{i-1})}{h}$$

$$\Rightarrow \; g_n'(t_{i-1}) + 4\, g_n'(t_i) + g_n'(t_{i+1}) = 3\left\{ g_n[t_i, t_{i+1}] + g_n[t_{i-1}, t_i] \right\}$$

$$= 3\, \frac{g_n(t_{i+1}) - g_n(t_{i-1})}{h} = \beta_i \,,$$

$$i = 1, 2, \cdots, n-1$$

$g_n'(t_i)$ : $n+1$ unknowns

$g_n(t_i) = f(t_i)$ , $i = 0, 1, \cdots, n$ : Given

## End Conditions:

1. Complete Cubic Spline Interpolation : $g_n'(a) = f'(a)$,
$g_n'(b) = f'(b)$.

System of equations

$$\begin{bmatrix} 4 & 1 & & & & 0 \\ 1 & 4 & 1 & & & \\ 0 & 1 & 4 & 1 & & \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix} \begin{bmatrix} g_n'(t_1) \\ g_n'(t_2) \\ \vdots \\ g_n'(t_{n-1}) \end{bmatrix} = \begin{bmatrix} \beta_1 - f'(a) \\ \beta_2 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} - f'(b) \end{bmatrix}$$

$$\beta_i = 3\,\frac{f(t_{i+1}) - f(t_{i-1})}{h}, \quad i = 1, \cdots, n-1$$

2) <u>Natural</u> <u>End</u> <u>Conditions</u> : $g_n''(a) = g_n''(b) = 0$.

For $x \in [a, t_1] = [t_0, t_1]$,

$$g_n''(a) = P_0''(t_0+)$$

$$= \frac{6\, g_n[t_0, t_1] - 4\, g_n'(t_0) - 2\, g_n'(t_1)}{h} = 0$$

$$\Rightarrow 2\, g_n'(t_0) + g_n'(t_1) = 3\, \frac{g_n(t_1) - g_n(t_0)}{h}$$

Similarly, $g''(b) = 0 \Rightarrow$

$$g_n'(t_{n-1}) + 2\, g_n'(t_n) = 3\, \frac{g_n(t_n) - g_n(t_{n-1})}{h}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & 4 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 4 & 1 & \cdots & & 0 \\ \vdots & & & & & & \\ & & & & & & 0 \\ & & & & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix} \begin{bmatrix} g_n'(t_0) \\ g_n'(t_1) \\ g_n'(t_2) \\ \vdots \\ g_n'(t_{n-1}) \\ g_n'(t_n) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$\beta_i = 3\,\frac{f(t_{i+1}) - f(t_{i-1})}{h}, \quad \beta_0 = 3 \cdot \frac{f(t_1) - f(t_0)}{h},$$

$$i = 1, \ldots, n-1$$

$$\beta_n = 3\,\frac{f(t_n) - f(t_{n-1})}{h}$$

$f \in C^1[a, b]$,

$g_n(t_i) = f(t_i)$, $i = 0, 1, \cdots, n$

$g_n'(t_i)$, $i = 0, 1, \cdots, n$ obtained by solving a tridiagonal system of linear equations.

## Order of Convergence.

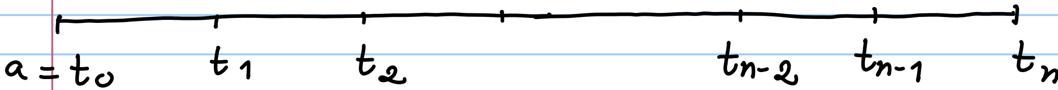
1) Complete Cubic Spline Interpolation:

$$\| f - g_n \|_\infty \le C h^4$$

2) Natural End Conditions:

$$\| f - g_n \|_\infty \le C h^2$$

## not - a - knot condition :




$$g_n'''(t_1-) = g_n'''(t_1+) \,, \quad g_n'''(t_{n-1}^-) = g_n'''(t_{n-1}+)$$

$g_n$ reduces to a single cubic polynomial in $[t_0, t_2]$ and in $[t_{n-2}, t_n]$

Order of convergence : $h^4$