Tutorial 1

Note Title 3/11/2011

Q.1. Let $x_0, x_1, \cdots, x_n$ be distinct points in $[a,b]$ and let $l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$: Lagrange polynomial.

Show that $\sum_{i=0}^{n} l_i(x) = 1$.

Solution : Consider $f(x) = 1$, $x \in [a,b]$.

Let p_n : polynomial of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n.$$

Then $1 = p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x) = \sum_{i=0}^{n} l_i(x)$

$$f(x) = x \implies p_n(x) = x, \quad n \geq 1$$

$$x = \sum_{i=0}^{n} f(x_i)\, l_i(x) = \sum_{i=0}^{n} x_i\, l_i(x)$$

Q.2. $f(x) = \frac{1}{x}$, $x \in [1,2]$, $x_0, x_1, \cdots, x_n \in [1,2]$

Show that $f[x_0, x_1, \cdots, x_n] = \frac{(-1)^n}{x_0 x_1 \cdots x_n}$ ... ①

Solution: $f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0}$

① is true for $n = 0$: $f[x_0] = f(x_0)$

Assume ① for $n = m-1$

$$f[x_0, x_1, \cdots, x_m] = \frac{(-1)^{m-1}\left\{\frac{1}{x_1 \cdots x_m} - \frac{1}{x_0 \cdots x_{m-1}}\right\}}{x_m - x_0}$$

$$= (-1)^m \left\{\frac{1}{x_0 x_1 \cdots x_m}\right\}$$

Q.3. $f(x) = 279x^4 + 44x^3 - 13x^2 + 47x + 23$.

Find the divided differences

i) $f[1, 2, 3, 4, 5] = 279 = \dfrac{f^{(4)}(c)}{4!}$

ii) $f[1, 2, 3, 4, 5, 6] = 0 = \dfrac{f^{(5)}(d)}{5!}$

Q.4. $f(x) = \ln(x)$, p_3 : polynomial of degree ≤ 3 interpolating f at $1, \frac{4}{3}, \frac{5}{3}, 2$. Find a lower bound for $\left|f\left(\frac{3}{2}\right) - p_3\left(\frac{3}{2}\right)\right|$.

Solution:

$$f(x) - p_3(x) = f\left[1, \tfrac{4}{3}, \tfrac{5}{3}, 2, x\right] \omega(x),$$

$$= \frac{f^{(4)}(c_x)}{4!} \omega(x),$$

where $\omega(x) = (x-1)\left(x-\frac{4}{3}\right)\left(x-\frac{5}{3}\right)(x-2)$, $x \in [1,2]$

$\omega\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \frac{1}{6}\left(-\frac{1}{6}\right)\left(-\frac{1}{2}\right) = \frac{1}{144}$, $\quad f^{(4)}(x) = -\frac{6}{x^4}$

$$\left|f\left(\tfrac{3}{2}\right) - p_3\left(\tfrac{3}{2}\right)\right| \ge \frac{6}{16 \times 144} = \frac{1}{384}$$

Q.5 $f(x_0) = f'(x_0) = 0$, $f(x_1) = f'(x_1) = f''(x_1) = 0$.

Find the polynomial of degree ≤ 5 which interpolates f at $x_0, x_0, x_1, x_1, x_1, x_2$

Solution : p_5 : double zero at x_0, triple zero at x_1

$$p_5(x) = \alpha (x - x_0)^2 (x - x_1)^3$$

$$f(x_2) = p_5(x_2) = \alpha (x_2 - x_0)^2 (x_2 - x_1)^3$$

$$\Rightarrow \quad \alpha = \frac{f(x_2)}{(x_2 - x_0)^2 (x_2 - x_1)^3}$$

Q.6. $f(x) = x^2, \ x \in [N, N+1]$.

$p_1(x)$: linear polynomial interpolating f at N and $N+1$.

Find the largest error.

Solution: $\|f - p_1\|_\infty \leq \dfrac{\|f''\|}{8} = \dfrac{1}{4}$

$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$: Power form.

$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$: Lagrange form.

$p_n(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + \cdots +$

$a_n (x - x_0) \cdots (x - x_{n-1})$: Newton form

$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots +$

$f[x_0, x_1, \ldots, x_n](x - x_0) \cdots (x - x_{n-1})$

$$f(x) = f(x_0) + f[x_0, x_1](x-x_0) + \cdots +$$

$$f[x_0, x_1, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$$

$$+ f[x_0, x_1, \cdots, x_n, x](x-x_0)\cdots(x-x_n) .$$

f : $n+1$ times differentiable,

$$x_0 = x_1 = \cdots = x_n$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+ \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$: Taylor's Theorem.

Note Title 3/10/2011

Riemann Integration

$f : [a, b] \longrightarrow \mathbb{R}$ continuous.

$a = t_0 < t_1 < \cdots < t_n = b$: uniform partition

$t_{i+1} - t_i = h = \frac{b-a}{n}$, $i = 0, 1, \cdots, n-1$

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} h \, f(t_i)$$

$$= \lim_{n \to \infty} (b-a) \frac{f(t_0) + f(t_1) + \cdots + f(t_{n-1})}{n} = \int_a^b f(x)\,dx$$

Fundamental Theorem of Integral Calculus

$f : [a, b] \longrightarrow \mathbb{R}$ continuous, $F : [a, b] \longrightarrow \mathbb{R}$ is such that $F'(x) = f(x)$, $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

F : anti-derivative of f.

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$F(x) = a_0 x + a_1 \frac{x^2}{2} + \cdots + a_n \frac{x^{n+1}}{n+1} \Longrightarrow F'(x) = f(x)$$

Numerical Integration

$f : [a,b] \longrightarrow \mathbb{R}$, $x_0, x_1, \cdots, x_n$: $n+1$ distinct points in $[a,b]$,

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$: Lagrange Polynomial

$$l_i(x_j) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

$$\int_a^b f(x)\,dx \simeq \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,dx$$

$$= \sum_{i=0}^{n} w_i\, f(x_i)$$

$$\int_a^b f(x)\,dx - \int_a^b p_n(x)\,dx = \int_a^b f[x_0, x_1, \cdots, x_n\ x]\,\omega(x)\,dx$$

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

$$\left| \int_a^b f(x)\,dx - \int_a^b p_n(x)\,dx \right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}(b-a)^{n+2}$$

Mean Value Theorem for Integrals

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $g : [a, b] \to \mathbb{R}$ be an integrable function such that either $g(x) \geq 0$ or $g(x) \leq 0$, $x \in [a,b]$. Then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

Proof: For the sake of definitness, let $g(x) \geq 0$.
Since f is continuous on $[a,b]$, f is bounded.
Let m and M denote respectively the absolute minimum and the absolute maximum of f. Then

$$m \leq f(x) \leq M, \quad x \in [a,b]$$

$$\Rightarrow m\, g(x) \leq f(x)\, g(x) \leq M\, g(x), \quad x \in [a,b]$$

$$\Rightarrow m \int_a^b g(x)\,dx \leq \int_a^b f(x) g(x)\,dx \leq M \int_a^b g(x)\,dx$$

$$m \int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b g(x)\,dx .$$

Case i) $\int_a^b g(x)\,dx = 0 \Rightarrow \int_a^b f(x)g(x)\,dx = 0$

$\Rightarrow \int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$ for any $c \in [a,b]$.

Case ii) $\int_a^b g(x)\,dx > 0 \Rightarrow$

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

By the Intermediate Value Theorem, $= f(c)$.