Note Title 4/30/2012

Assignment 1 : Solution.

Q.1 $f : [a, b] \longrightarrow \mathbb{R}$ continuous, $x_0, x_1, \cdots, x_n \in [a, b]$,

$\alpha_1, \cdots, \alpha_n$: non-negative numbers.

Let $m = \min\limits_{x \in [a,b]} f(x)$, $M = \max\limits_{x \in [a,b]} f(x)$

For $i = 1, \cdots, n$, $m \leq f(x_i) \leq M$

$\Rightarrow \alpha_i m \leq \alpha_i f(x_i) \leq M \alpha_i$

$\Rightarrow m \sum\limits_{i=1}^{n} \alpha_i \leq \sum\limits_{i=1}^{n} \alpha_i f(x_i) \leq M \sum\limits_{i=1}^{n} \alpha_i$

We have

$$\Rightarrow \quad m \sum_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \alpha_i f(x_i) \leq M \sum_{i=1}^{n} \alpha_i$$

If $\sum_{i=1}^{n} \alpha_i = 0$, then $\sum_{i=1}^{n} \alpha_i f(x_i) = 0$ and

$$\sum_{i=1}^{n} \alpha_i f(x_i) = f(c) \sum_{i=1}^{n} \alpha_i \quad \text{for } \underline{\text{any}} \ c \in [a,b]$$

If $\sum_{i=1}^{n} \alpha_i > 0$, then $m \leq \dfrac{\sum_{i=1}^{n} \alpha_i f(x_i)}{\sum_{i=1}^{n} \alpha_i f(x_i)} \leq M$

and the result follows by the Intermediate Value Theorem.

Q.2 $f : [a,b] \longrightarrow \mathbb{R}$ continuous,

$g : [a,b] \longrightarrow \mathbb{R}$ integrable, $g(x) \leq 0$, $x \in [a,b]$

Let $m = \min\limits_{x\in[a,b]} f(x)$, $M = \max\limits_{x\in[a,b]} f(x)$

For $x \in [a,b]$, $m \leq f(x) \leq M$

$\Rightarrow m\, g(x) \geq f(x) g(x) \geq M\, g(x)$

$\Rightarrow m \int_a^b g(x)\,dx \geq \int_a^b f(x) g(x)\,dx \geq M \int_a^b g(x)\,dx$

We have

$$\Rightarrow m \int_a^b g(x)\,dx \geq \int_a^b f(x)g(x)\,dx \geq M \int_a^b g(x)\,dx$$

If $\int_a^b g(x)\,dx = 0$, then $\int_a^b f(x)g(x)\,dx = 0$ and any c $\in [a,b]$ will do.

If $\int_a^b g(x)\,dx < 0$, then

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M.$$

Use Intermediate Value Theorem

Q.3 $f(x) = x^3 - 2x^2 + x$

$f(x) - \frac{1}{2} = x^3 - 2x^2 + x - 1/2 = g(x)$

$g(0) = -1/2$, $g(2) = \frac{3}{2}$, g continuous.

By the Intermediate Value Theorem,

$g(c) = 0$ for some c.

$\Rightarrow f(c) = \frac{1}{2}$

Q.4 Let $N \le M < N+1$. Then $r = \dfrac{M}{n+1} < 1$.

For $n > N$,

$$0 \le \frac{M^n}{n!} = \frac{M^N \, M^{n-N}}{N! \, (N+1) \cdots n} \le \frac{M^N}{N!} (r)^{n-N}$$

Constant $\quad 0$ as $n \to \infty$.

Q.5 $l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$, $l_i(x_k) = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$

$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$ is the unique polynomial of degree $\leq n$ interpolating f at $x_0, x_1, \cdots, x_n$.

Consider $f(x) = x$. Then $p_n(x) = f(x)$ for $n \geq 1$

Thus $\sum_{i=0}^{n} x_i\, l_i(x) = x$.