Note Title 4/30/2012

Assignment 4 : Solution

Q-1 a)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & & \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

$A = A^T \Rightarrow a_{ij} = a_{ji}$

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$$

$$= a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} = a_{ji}^{(1)} \Rightarrow A_{22} : \text{symmetric}$$

$$\det(A_k) = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22}^{(1)} & \cdots & a_{2k}^{(1)} \\ \vdots & & & \\ 0 & a_{k2}^{(1)} & \cdots & a_{kk}^{(1)} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & B_{12} \\ 0 & B_{k-1} \end{bmatrix} = a_{11} \det(B_{k-1})$$

$\det(A_k) > 0$, $a_{11} > 0 \Rightarrow \det(B_{k-1}) > 0$

$\Rightarrow A_{22}$ positive-definite.

b) Given $\sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \leq |a_{jj}|, \quad j = 1, \cdots, n$

Consider $\sum_{\substack{i=2 \\ i \neq j}}^{n} |a_{ij}^{(1)}| \leq \sum_{\substack{i=2 \\ i \neq j}}^{n} |a_{ij}| + \frac{|a_{1j}|}{|a_{11}|} \sum_{\substack{i=2 \\ i \neq j}}^{n} |a_{i1}|$

$$\leq |a_{jj}| - |a_{1j}| + \frac{|a_{1j}|}{|a_{11}|}\left(|a_{11}| - |a_{j1}|\right)$$

$$= |a_{jj}| - \frac{|a_{1j}|\,|a_{j1}|}{|a_{11}|} \leq \left| a_{jj} - \frac{a_{1j}\, a_{j1}}{a_{11}} \right| = |a_{jj}^{(1)}|$$

Q.2 $g(x) = x(2-x)$

$g(x) = x \iff x = 0$ or $x = 1$

$x_{n+1} = g(x_n) = x_n(2-x_n)$

$x_{n+1} - 1 = -1 + 2x_n - x_n^2 = -(x_n - 1)^2$

$|x_{n+1} - 1| = |x_n - 1|^2 = |x_{n-1} - 1|^{2^2} = \cdots$

$= |x_0 - 1|^{2^{n+1}}$

$x_{n+1} \longrightarrow 1 \iff |x_0 - 1| < 1 \iff x_0 \in (0,2)$

$g(x) = x(2-x) : [0,2] \longrightarrow [0,2]$

g : continuous

$(0,2)$: largest interval such that

$x_0 \in (0,2) \Longrightarrow x_n \longrightarrow 1$.

$g'(x) = 2 - 2x = -2(x-1)$

$|g'(x)| = 2|x-1| \geq 1 \Longleftrightarrow |x-1| \geq 1/2$

$\Longleftrightarrow x \in (0, \frac{1}{2}] \cup [\frac{3}{2}, 1)$: Sufficient condition for convergence not satisfied

Q.3 $y'(x) = -2y(x), \quad 0 \le x \le 1, \quad y(0) = 1.$

a) Exact Solution : $y(x) = e^{-2x}, \quad y'(x) = -2e^{-2x},$

$y''(x) = 4e^{-2x}, \quad |y''(x)| \le Y = 4$

$f(x,y) = -2y, \quad f_y(x,y) = -2, \quad |f_y(x,y)| \le L = 2$

$$|e_n| \le \frac{hY}{2L}\left(e^{(x_n - x_0)L} - 1\right) = \frac{4h}{4}(e^2 - 1)$$

$$= h(e^2 - 1)$$

b) $y_{n+1} = y_n + h f(x_n, y_n)$

$= (1 - 2h)\, y_n \quad , \quad y(0) = 1 \Rightarrow y_n = (1 - 2h)^n$

c) $y_{10} = (1-0.2)^{10} = (0.8)^{10} \simeq 0.107374$

exact solution : $y(1) = e^{-2} \simeq 0.135335$

error : $y(1) - y_{10} = 0.027961$

Bound : $0.1\,(e^{2}-1)$

Q.4 $y' = y^{\alpha}$: $\alpha < 1$, $y(0) = 0$

Exact Solution: $\dfrac{y^{\alpha+1}}{\alpha+1}$

Euler Method: $y_{n+1} = y_n + h\, f(x_n, y_n)$

$$= y_n + h\, y_n^{\alpha}$$

$y_0 = 0 \Rightarrow y_n = 0$ for all n

Rungé-Kutta Method of Order 2:

$y' = y^{\alpha}, \quad y(0) = 0$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2),$$

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f(x_n + h, y_n + k_1)$$

$\Rightarrow y_n = 0$ for all n