Note Title 5/1/2012

Assignment 5 : Solution

Q.1 $y' = f(x, y)$, $y(x_0) = y_0$, $x_{n+1} - x_n = h$

$$y(x_{n+1}) - y(x_{n-1}) = \int_{x_{n-1}}^{x_{n+1}} f(x, y(x))\, dx$$

Apply Simpson Rule:

$$y_{n+1} - y_{n-1} = \frac{2h}{6}\Big[f(x_{n-1}, y_{n-1}) + 4 f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big]$$

First Iterate:

$$y_{n+1}^{(1)} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4 f_n + f_{n+1}^{(0)}\right)$$

error $= -\frac{h^5}{90} y^{(5)}(\xi)$

Q.2 $y' = f(x, y)\ ,\ y(x_0) = y_0$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p_1(x)\,dx\ ,$$

p_1 : linear polynomial on $[x_n, x_{n+1}]$

interpolating $f(x, y(x))$ at x_n and x_{n+1}.

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right] \quad \cdots (*)$$

$$y(x_{n+1}) \simeq y(x_n) + h\,y'(x_n)$$

$$= y(x_n) + h\,f(x_n, y(x_0))$$

$y(x_n) \simeq y_n$, $hf(x_n, y(x_n)) \simeq h\, f(x_n, y_n) = k_1$

$h\, f(x_{n+1}, y(x_{n+1})) \simeq hf(x_n + h, y_n + k_1) = k_2$

Then $(*)$ becomes Rungé - Kutta method of order 2 .

Q.3 Note that

$$Ax(1) = x_2, \quad Ax(i) = x_{i-1} + x_{i+1}, \quad 2 \le i \le n-1,$$

$$Ax(n) = x_{n-1}.$$

$$Au^{(k)}(1) = \sin \frac{2k\pi}{n+1} = 2\cos\frac{k\pi}{n+1} \sin\frac{k\pi}{n+1},$$

$$Au^{(k)}(i) = \sin\frac{(i-1)k\pi}{n+1} + \sin\frac{(i+1)k\pi}{n+1}$$

$$= 2\cos\frac{k\pi}{n+1} \sin\frac{ik\pi}{n+1}, \quad 2 \le i \le n-1,$$

$$A u^{(k)}(n) = \frac{\sin (n-1) k\pi}{n+1}$$

$$= 2 \cos \frac{k\pi}{n+1} \sin \frac{n k\pi}{n+1}$$

Thus $A u^{(k)} = \left(2 \cos \frac{k\pi}{n+1} \right) u^{(k)}$

and the eigenvalues of A are

$$2 \cos \frac{k\pi}{n+1}, \quad k = 1, 2, \cdots, n$$

Q.4

$$A = \begin{bmatrix} d & e & 0 & \cdots & \\ e & d & e & 0 & \cdots \\ 0 & e & d & e & \cdots \\ & & \ddots & \ddots & \ddots \\ & 0 & & e & d & e \\ & & & & e & d \end{bmatrix}$$

$$= d I + e \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & 0 \\ & & \ddots & & \\ & 0 & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}$$

eigenvalues of A are :

$$d + 2e \cos \frac{k\pi}{n+1}, \quad k = 1, \cdots, n$$