Examination 1 : Solution

Note Title 2/17/2011

1. Define $l_i(x) = \dfrac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_k)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_k)}$, $i = 0, 1, \ldots, k$

$$l_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Then $p_k(x) = \sum_{i=0}^{k} f(x_i)\, l_i(x)$ is a polynomial of degree $\leq k$ and

$p_k(x_j) = f(x_j)$, $j = 0, 1, \ldots, k$.

If q_k is another interpolating polynomial of degree $\leq k$, then

$$(p_k - q_k)(x_j) = 0, \quad j = 0, 1, \ldots, k$$

and hence $p_k(x) = q_k(x)$.

2. Let p_{k-1} and q_{k-1} be polynomials of degree $\leq k-1$ such that $p_{k-1}(x_j) = f(x_j)$, $\underline{j = 0, 1, \cdots, k-1}$,

$q_{k-1}(x_j) = f(x_j)$, $\underline{j = 1, 2, \cdots, k}$.

Consider $p_k(x) = \dfrac{(x_k - x)\, p_{k-1}(x) + (x - x_0)\, q_{k-1}(x)}{x_k - x_0}$.

Then p_k : polynomial of degree $\leq k$,

$p_k(x_j) = f(x_j)$, $j = 0, 1, \cdots, k$ and

coeff. of x^k in p_k $= \dfrac{\text{coeff. of } x^{k-1} \text{ in } q_{k-1} - \text{coeff. of } x^{k-1} \text{ in } p_{k-1}}{x_k - x_0}$

3. $f : [a,b] \to \mathbb{R}$ twice continuously differentiable

$$f(x) = f(a) + f[a,b](x-a) + f[a\ b\ x](x-a)(x-b)$$

$$\int_a^b f(x)\,dx = f(a)(b-a) + f[a\ b]\frac{(b-a)^2}{2}$$

$$+ \int_a^b f[a\ b\ x]\,(x-a)\,(x-b)\,dx$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) + f[a\ b\ c]\int_a^b (x-a)(x-b)\,dx$$

(since $f[a\ b\ x]$ is continuous and $(x-a)(x-b) \le 0$)

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) + \frac{f''(\alpha)}{2}\left(-\frac{(b-a)^3}{6}\right),$$

Rule

error.

for some $\alpha \in (a,b)$

4 $\int_0^1 f(x)\,dx \simeq \frac{1}{8}\left(f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right)$

$f(x) = 1$. $LHS = 1$, $RHS = \frac{1}{8}(1+3+3+1) = 1$

$f(x) = x$. $LHS = \frac{1}{2}$, $RHS = \frac{1}{8}(1+2+1) = \frac{1}{2}$

$f(x) = x^2$. $LHS = \frac{1}{3}$, $RHS = \frac{1}{8}\left(\frac{1}{3} + \frac{4}{3} + 1\right) = \frac{1}{3}$

$f(x) = x^3$. $LHS = \frac{1}{4}$, $RHS = \frac{1}{8}\left(\frac{1}{9} + \frac{8}{9} + 1\right) = \frac{1}{4}$

$f(x) = x^4$. $LHS = \frac{1}{5}$, $RHS = \frac{1}{8}\left(\frac{1}{27} + \frac{16}{27} + 1\right) = \frac{11}{54}$

$LHS \neq RHS$.

The rule is exact for polynomials of degree ≤ 3.

5. $\int_a^b f(x)\,dx = T_n + \frac{h^2}{12}\left(f'(a) - f'(b)\right) + O(h^4)$

$\int_a^b f(x)\,dx = T_{\frac{n}{2}} + \frac{(2h)^2}{12}\left(f'(a) - f'(b)\right) + O(h^4)$

$T_n^1 = \frac{4T_n - T_{\frac{n}{2}}}{3}$. $\int_a^b f(x)\,dx = T_n^1 + O(h^4)$

$T_n = \frac{h}{2}\left(f(a) + f(b)\right) + h\sum_{i=1}^{n-1} f(t_i)$

$T_{\frac{n}{2}} = h\left(f(a) + f(b)\right) + 2h\sum_{\substack{i=2 \\ i \text{ even}}}^{n-1} f(t_i)$

Simpson rule

$T_n^1 = \frac{(2h)}{6}\left(f(a) + f(b)\right) + \frac{4(2h)}{6}\sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f(t_i) + \frac{2(2h)}{6}\sum_{\substack{i=1 \\ i \text{ even}}}^{n-1} f(t_i)$ ..

6. $g(x) = f[x_0\ x] = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 . \end{cases}$

For $x \neq x_0$,

$$g'(x) = \frac{(x-x_0) f'(x) - [f(x) - f(x_0)]}{(x-x_0)^2} = \frac{f[x\ x] - f[x_0\ x]}{x - x_0} = f[x_0\ x\ x].$$

$$g'(x_0) = \lim_{h \to 0} \frac{f[x_0, x_0+h] - f'(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - h f'(x_0)}{h^2} = \lim_{h \to 0} \frac{\frac{h^2}{2} f''(c)}{h^2},$$ by the extended MVT

$c \in [x_0\ x_0+h]$

$$= \frac{f''(x_0)}{2} = f[x_0\ x_0\ x_0]$$

7. $[A:b] = \left[\begin{array}{ccccccc|c} a_{11} & a_{12} & 0 & \cdots & & & 0 & b_1 \\ a_{21} & a_{22} & a_{23} & & & & & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & & & & \vdots \\ \vdots & & 0 & & & & & \vdots \\ & & & & & a_{n,n-1} & a_{nn} & b_n \end{array}\right]$

1st step :

$m_{21} = \dfrac{a_{21}}{a_{11}}$, $R_2 - m_{21} R_1$: $\left.\begin{array}{l} a_{22}^{(1)} = a_{22} - m_{21} a_{12} \\ b_2^{(1)} = b_2 - m_{21} b_1 \end{array}\right\}$ 3 mult./div. + 2 subtractions.

$[A:b] = \left[\begin{array}{cccccccc} a_{11} & a_{12} & 0 & \cdots & & 0 & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & 0 & \cdots & 0 & b_2^{(1)} \\ \vdots & & & & & & \\ \vdots & & & & & & \\ 0 & 0 & \cdots & & a_{n,n-1} & a_{nn} & b_n \end{array}\right]$

Each step : 3 mult./div. + 2 sub.

Total : $3(n-1)$ mult./div. + $2(n-1)$ sub.

Back Substitution:

$$u_{11} x_1 + u_{12} x_2 = y_1 .$$

$$u_{22} x_2 + u_{23} x_3 = y_2$$

$$\vdots$$

$$u_{n-1,n-1} x_{n-1} + u_{n-1,n} x_n = y_{n-1}$$

$$u_{n,n} x_n = y_n .$$

$$x_n = \frac{y_n}{u_{nn}}, \quad x_i = \frac{y_i - u_{i,i+1} x_{i+1}}{u_{ii}}, \quad i = n-1, \cdots, 1$$

Total : $n-1$ mult. + $n-1$ subtractions + n divisions

8. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{bmatrix} \xrightarrow[\substack{R_3-3R_1 \\ R_4-4R_1}]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 8 \\ 0 & 3 & 8 & 14 \end{bmatrix} \xrightarrow{\substack{R_3-2R_2 \\ R_4-3R_2}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_4-2R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

9. $A = LU \Rightarrow \det(A) = \det(L)\det(U)$

$$= u_{11} u_{22} \cdots u_{nn} \neq 0$$

$\Rightarrow u_{ii} \neq 0, \quad i = 1, 2, \cdots, n$

For $1 \le i, j \le k$, $A_k(i,j) = \sum_{p=1}^{n} L(i,p)\, U(p,j)$

$$= \sum_{p=1}^{k} L(i,p)\, U(p,j) = \sum_{p=1}^{k} L_k(i,p)\, U_k(p,j)$$

$$= \left(L_k U_k\right)(i,j)$$

$A_k = L_k U_k \Rightarrow \det(A_k) = \det(U_k)$

$$= u_{11} \cdots u_{kk} \neq 0 .$$

10. $A = MM^T \Rightarrow A^T = (M^T)^T M^T = MM^T = A$

Let $x \neq \overline{0}$. Since M^T is invertible, $y = M^T x \neq 0$.

Consider $x^T A x = x^T M M^T x = y^T y$

$$= y_1^2 + \cdots + y_n^2 > 0$$

$\Rightarrow$ A is positive-definite.