Examination 2 : Solution

Note Title 4/20/2011

Q.1. p_k : poly. of degree $\le k$, $p_k(x_j) = f(x_j)$, $j = 0, \cdots, k$

The coefficient of x^k in $p_k(x) = f[x_0, \cdots, x_k]$

$e(x) = f(x) - p_k(x)$

$e(x_j) = 0$, $j = 0, 1, \cdots, k \Rightarrow e'$ has at least k zeroes

$\Rightarrow e^{(k)}$ has at least one zero, say c.

$$0 = e^{(k)}(c) = f^{(k)}(c) - p_k^{(k)}(c)$$
$$= f^{(k)}(c) - k!\, f[x_0, \cdots, x_k]$$

Q.2 $f(x) = 1 : \quad 2 = w_0 + w_1 + w_2 \quad \cdots (a)$

$f(x) = x : \quad 0 = -w_0 + w_1 x_1 + w_2 \quad \cdots (b)$

$f(x) = x^2 : \quad \frac{2}{3} = w_0 + w_1 x_1^2 + w_2 \quad \cdots (c)$

$f(x) = x^3 : \quad 0 = -w_0 + w_1 x_1^3 + w_2 \quad \cdots (d)$

$(d) - (b) \Rightarrow w_1 x_1 (x_1^2 - 1) = 0$

(a) and (c) $\Rightarrow w_1 \neq 0 , \quad x_1^2 \neq 1$

Hence $x_1 = 0$.

$(b) \Rightarrow w_2 = w_0 , \quad (c) \Rightarrow w_2 = w_0 = \frac{1}{3} , \quad (a) \Rightarrow w_1 = \frac{4}{3}$

Q.3 $\quad Ax = b$

$$x = x_1 e_1 + \cdots + x_n e_n$$

$$Ax = x_1 A e_1 + \cdots + x_j A e_j + \cdots + x_n A e_n$$

$$= x_1 C_1 + \cdots + x_j C_j + \cdots + x_n C_n = b$$

$$= x_1 C_1 + \cdots + \left(\frac{1}{\alpha} x_j\right)(\alpha C_j) + \cdots + x_n C_n$$

Q.4 $\quad (Ax)(i) = \sum_{j=1}^{n} a_{ij} x_j$

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| |x_j|$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}| |x_j| \leq \left(\max_j \sum_{i=1}^{n} |a_{ij}| \right) \|x\|_1$$

$$\Rightarrow \|A\|_1 \leq \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ij_0}|$$

$$\sum_{i=1}^{n} |a_{ij_0}| = \|A e_{j_0}\|_1 \leq \|A\|_1$$

Q.5 $f(x) = f(a-h) + f[a-h, a+h](x-a+h)$

$+ f[a-h, a+h, x](x-a+h)(x-a-h)$

$f'(x) = f[a-h, a+h] + f[a-h, a+h, x\ x]$
$(x-a+h)(x-a-h)$

$+ f[a-h, a+h, x]\{x-a-h + x-a+h\}$
$f'(a) = f[a-h, a+h] + f[a-h, a+h, a, a](-h^2)$

Q. 6 $\quad Ax = \lambda x \implies \|Ax\|_1 = |\lambda| \, \|x\|_1$

$$\implies |\lambda| \le \|A\|_1$$

Hence $5 \le \|A\|_1$

The evs of A^{-1} are $3, 2, 1, \frac{1}{5}$

Hence $3 \le \|A^{-1}\|_1$

$\implies 15 \le \|A\|_1 \, \|A^{-1}\|_1$

Q. 7 $\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \cdots, n$

Jacobi Method:

$$x_i^{(k)} = \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k-1)} \right) / a_{ii}$$

$$x_i - x_i^{(k)} = - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_{ij}}{a_{ii}} \left(x_j - x_j^{(k-1)} \right)$$

$$|x_i - x_i^{(k)}| \leq \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \right) \| x - x^{(k-1)} \|_\infty$$

$$\| x - x^{(k)} \|_\infty \leq \mu \, \| x - x^{(k-1)} \|_\infty \leq \cdots \leq \mu^k \| x - x^{(0)} \|_\infty$$

Q.8 Gauss-Seidel Iterates : $x^{(0)} = \overline{0}$

$$x_1^{(1)} = \frac{3}{4}, \quad x_2^{(1)} = \frac{2 + x_1^{(1)} + x_2^{(0)}}{4} = \frac{11}{16}$$

$\left(\frac{1}{2}\right)$

$$x_3^{(1)} = \frac{2 + x_2^{(1)} + x_4^{(0)}}{4} = \frac{2 + \frac{11}{16}}{4} = \frac{43}{64}$$

$$x_4^{(1)} = \frac{3 + x_3^{(1)}}{4} = \frac{3 + \frac{43}{64}}{4} = \frac{235}{256}$$

Q. 9 By Gerschgorin Theorem, the evs of

A are $C \{z \in \mathbb{C} : |z-4| \leq 2\}$

$\cup \{z \in \mathbb{C} : |z-4| \leq 1\}$

A : real symmetric $\Rightarrow$ evs are real

evs of $A = \{x \in \mathbb{R} : |x-4| \leq 2\}$

$= [2, 6]$

Q. 10 $\quad y' = f(x, y), \quad y(a) = y_0$

$a = x_0 < x_1 < \cdots < x_N = b, \quad h = \frac{b-a}{N}$

<u>Euler's</u> <u>Method</u>

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, 2, \ldots .$$

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(c_n)$$

$$= y(x_n) + h\, f(x_n, y(x_n)) + \frac{h^2}{2} y''(c_n)$$

local discretization error : $y(x_n) = y_n$

Assumption

$$y(x_{n+1}) - y_{n+1} = \frac{h^2}{2} y''(c_n)$$

Midpoint Rule

$$y_{n+1} = y_{n-1} + 2h\, f(x_n, y_n), \quad n = 1, 2, \ldots$$

$$y(x_{n+1}) - y(x_{n-1}) = 2h\, f(x_n, y(x_n)) + \frac{h^3}{3} y'''(d_n)$$

local discretizion error :

$$y(x_{n+1}) - y_{n+1} = \frac{h^3}{3} y'''(d_n)$$

Q.11 $x_{n+1} - x^* = g(x_n) - g(x^*)$

$= g'(c_n)(x_n - x^*)$

$\Rightarrow |x_{n+1} - x^*| \leq M \, |x_n - x^*|$

$= M \, |x_n - x_{n+1} + x_{n+1} - x^*|$

$\leq M \, |x_{n+1} - x_n| + M \, |x_{n+1} - x^*|$

$\Rightarrow |x_{n+1} - x^*| \leq \frac{M}{1-M} |x_{n+1} - x_n|$

Q. 12 $\quad A$ positive-definite $\Rightarrow \lambda_1 > 0$

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, \quad \alpha_1 \neq 0$$

$$A^j x = \alpha_1 \lambda_1^j u_1 + \alpha_2 \lambda_2^j u_2 + \cdots + \alpha_n \lambda_n^j u_n$$

$$= \lambda_1^j \left(\alpha_1 u_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j u_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^j u_n \right)$$

$$\frac{A^j x}{\|A^j x\|} = \frac{\alpha_1 u_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j u_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^j u_n}{\|\alpha_1 u_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j u_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^j u_n\|}$$

$$\longrightarrow \frac{\alpha_1 u_1}{\|\alpha_1 u_1\|}$$

Q. 13 $\quad Q = [q_1 \; q_2 \cdots q_n]$

$$Q^t Q = \begin{bmatrix} q_1^t \\ q_2^t \\ \vdots \\ q_n^t \end{bmatrix} [q_1 \; q_2 \cdots q_n] = [q_i^t q_j] = I$$

$$\Rightarrow \quad q_i^t q_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Q. 14 $A = QR$

$$Q^t A Q = RQ = A_1$$

$$\det(A_1 - xI) = \det(Q^t(A - xI)Q)$$
$$= \det(Q^t)\det(Q)\det(A - xI)$$
$$= \det(A - xI)$$