Note Title 11/27/2010

Gauss elimination with partial pivoting

$$Ax = b$$

Assumption : A is invertible ; $\det(A) \neq 0$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We consider the elements in the first column and let $|a_{k1}| = \max\limits_{1 \leq i \leq n} |a_{i1}|$.

Then since A is invertible, $a_{k1} \neq 0$

Interchange the first and the k th row.

Interchange the first and the k-th row :

$$\begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_k \\ b_2 \\ \vdots \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Define $\quad m_{i1} = \dfrac{a_{i1}}{a_{k1}}, \quad i \neq k, \quad i = 1, \cdots, n$

$$\tilde{R}_i \longrightarrow \tilde{R}_i - m_{i1} \tilde{R}_1$$

Note that

$|m_{i1}| \leq 1$

$$
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{21} & \cdots & \tilde{a}_{n1} \\
0 & \tilde{a}_{22}^{(1)} & \cdots & \tilde{a}_{2n}^{(1)} \\
\vdots & & & \\
0 & \tilde{a}_{n2}^{(1)} & \cdots & \tilde{a}_{nn}^{(1)}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2^{(1)} \\ \vdots \\ \tilde{b}_n^{(1)} \end{bmatrix}
$$

Let $\quad |\tilde{a}_{k2}^{(1)}| = \max\limits_{2 \le i \le n} |\tilde{a}_{i2}^{(1)}| \neq 0$ (why?)

Interchange 2nd and kth equation, define

$$
m_{i2} = \frac{\tilde{a}_{i2}^{(1)}}{\tilde{a}_{k2}^{(1)}}, \quad i \neq k, \quad i = 2, \cdots, n \quad \text{and perform}
$$

Continue...

$$
\bar{R}_i \longrightarrow \bar{R}_i - m_{i2} \bar{R}_2, \quad i = 3, \cdots, n
$$

$$A = [a_{ij}] \ : \ n \times n \ \text{matrix}, \quad e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow j\text{th place} \\ \\ \\ \\ \end{matrix}$$

$$A e_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = C_j$$

$$e_i^T A = [a_{i1} \ a_{i2} \ \cdots \ a_{in}] = R_i$$

$$A = [Ae_1 \quad Ae_2 \quad \cdots \quad Ae_n]$$

$$B = [Be_1 \quad Be_2 \quad \cdots \quad Be_n] = [C_1 \quad C_2 \quad \cdots \quad C_n]$$

$$AB = [ABe_1 \quad ABe_2 \quad \cdots \quad ABe_n]$$

$$= [AC_1 \quad AC_2 \quad \cdots \quad AC_n]$$

We have seen before that the operation

$$R_i \longrightarrow R_i - m_{i1} R_1 \qquad a_{ij}^{(1)} = a_{ij} - m_{i1} a_{1j}$$

can be performed by premultiplying A by an elementary matrix.

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ -m_{n1} & 0 & & 1 \end{bmatrix}}_{E^{(1)}} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A^{(1)}$$

Note that

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{I} \xrightarrow{R_i - m_{i1} R_1} \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \cdots & 0 \\ \vdots & & & \\ -m_{n1} & 0 & \cdots & 1 \end{bmatrix}}_{E^{(1)}}$$

We now show that the interchange of two rows of a matrix is equivalent to premultiplying by a permutation matrix obtained by interchanging the corresponding rows of the Identity matrix.

Perform $R_1 \leftrightarrow R_k$ on the Identity matrix to obtain

$$P_1 = \begin{array}{r c} & \qquad\qquad\quad\ \ \downarrow k \\ \begin{matrix} \\ \\ \\ \\ k \rightarrow \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & & & & 1 & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ 1 & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \end{array}$$

Then $P_1^2 = I$. Permutation matrix

Interchange of the first and the kth column.

$$P_1 = P_1^T = [e_k \; e_2 \cdots e_{k-1} \; e_1 \; e_{k+1} \cdots e_n]$$

$$P_1 = [e_k \ \ e_2 \cdots e_{k-1} \ \ e_1 \ \ e_{k+1} \cdots e_n]$$

$$AP_1 = [Ae_k \ \ Ae_2 \cdots Ae_{k-1} \ \ Ae_1 \ \ Ae_{k+1} \cdots Ae_n]$$

$$= [C_k \ \ C_2 \cdots C_{k-1} \ \ C_1 \ \ C_{k+1} \cdots C_n]$$

AP_1 : Interchange of the 1st and the kth column of A

$e_i^T A = R_i$: ith row of A

$$P_1 A = \begin{bmatrix} e_k^T \\ e_2^T \\ \vdots \\ e_{k-1}^T \\ e_1^T \\ \vdots \\ e_n^T \end{bmatrix} A = \begin{bmatrix} e_k^T A \\ e_2^T A \\ \vdots \\ e_{k-1}^T A \\ e_1^T A \\ \vdots \\ e_n^T A \end{bmatrix} = \begin{bmatrix} R_k \\ R_2 \\ \vdots \\ R_{k-1} \\ R_1 \\ \vdots \\ R_n \end{bmatrix}$$

: Interchange of the 1st and the kth row.

$I \quad : \quad R_1 \leftrightarrow R_k \quad (C_1 \leftrightarrow C_k) \; : \quad P_1$

$P_1^T = P_1 \, , \quad P_1^2 = I$

$P_1 A$: Interchange of the 1st and the kth row

$A P_1$: Interchange of the 1st and the kth column

Thus reduction of A to an upper triangular matrix U using partial pivoting can be represented as

$$E_{n-1} P_{n-1} E_{n-2} P_{n-2} \cdots E_1 P_1 A = U,$$

where P_i s are permutation matrices.

The above equation can be rewritten as

$$E'_{n-1} E'_{n-2} \cdots E'_1 \underbrace{P_{n-1} P_{n-2} \cdots P_1}_{\overset{\shortparallel}{P}} A = U.$$

Consider $E_2 P_2 E_1 P_1 A = U \quad (n=3)$.

Note that $P_2^2 = I$.

Hence $U = E_2 \underbrace{P_2 E_1 P_2} P_2 P_1 A$

Define

$$E_1' = P_2 E_1 P_2$$

Then

$$U = E_2 E_1' P_2 P_1 A$$

We now show that E_1' is also unit lower triangular

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$P_2 E_1$: Interchange of 2nd & 3rd row of E_1

$$P_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{31} & 0 & 1 \\ -m_{21} & 1 & 0 \end{bmatrix}$$

$(P_2 E_1) P_2$: Interchange of the second and third columns of $P_2 E_1$

$$E_1' = P_2 E_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{31} & 1 & 0 \\ -m_{21} & 0 & 1 \end{bmatrix}$$

: unit lower triangular

Gauss elimination with partial pivoting

$$PA = LU,$$

where P : permutation matrix

L : unit lower triangular matrix

U : upper triangular matrix.

$\det(P) = \pm 1$, $\det(L) = 1$, $\det(U) = \pm \det(A)$

$|l_{ij}| \leq 1$

Suppose that $A = [a_{ij}]$ is diagonally dominant by columns :

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \leq |a_{jj}| \quad , \quad j = 1, \cdots, n .$$

$$\Rightarrow \quad \sum_{i=2}^{n} |a_{i1}| \leq |a_{11}|$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

No need of row interchanges

Inverse of a matrix.

Classical formula : $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

Calculation of determinant in the classical way :

more than $n!$ flops.

$n!$ grows much faster than n^3.

$AX = I$: $A[x_1, x_2, \cdots x_n] = [e_1 \; e_2 \cdots e_n]$

Solve $\quad A x_i = e_i \;, \; i = 1, \cdots, n$.

Cost : LU decomposition (once) : $\frac{n^3}{3}$ flops

forward & backward substitution (n times) : $n^2 \times n = n^3$

Calculation of the determinant

$$PA = LU \implies \det(A) = \pm \det(U)$$

sign depends on the number of row interchanges

Cost : $\frac{n^3}{3}$ flops

Complete Pivoting

$Ax = b$, A : invertible .

First step : Let $|a_{kl}| = \max_{1 \le i,j \le n} |a_{ij}|$. n^2 comparisons

Interchange the 1st and kth row and the 1st and lth column.

$\tilde{A} = P_1 A Q_1$: Perform Gauss elimination to obtain

$$\tilde{A}^{(1)} = \begin{bmatrix} a_{kl} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$E_1 \tilde{A} = \tilde{A}^{(1)}$$

$$\parallel$$

$$E_1 P_1 A Q_1 = \tilde{A}^{(1)}$$

2nd Step: $\tilde{A}_{22} = \left[\tilde{a}^{(1)}_{ij} \right] : 2 \le i, j \le n$

Let $\left| \tilde{a}^{(1)}_{kl} \right| = \max\limits_{2 \le i,j \le n} \left| \tilde{a}^{(1)}_{ij} \right|$

Interchange 2nd and kth row and 2nd and lth column of $\tilde{A}^{(1)}$ and introduce zeros in the second column below diagonal

$$E_2 P_2 \tilde{A}^{(1)} Q_2 = \tilde{A}^{(2)}$$

$$E_2 P_2 E_1 P_1 A Q_1 Q_2 = \tilde{A}^{(2)}$$

Complete Pivoting

$$PAQ = LU,$$

P, Q : permutation matrices,

L : unit lower triangular

U : upper triangular

n^3 comparisons : expensive