Prof. Rana
lec - 2 (8-11-10)

Lecture – II

IKR

8/11/10

$\mathcal{C}$ – semi-algebra X

(i) $\phi, X \in \mathcal{C}$

(ii) $A, B \in \mathcal{C} \Longrightarrow A \cap B \in \mathcal{C}$

(iii) $A \in \mathcal{C} \Longrightarrow A^c = \bigsqcup_{i=1}^{n} C_i, \; C_i \in \mathcal{C}$

$C_i \cap C_j = \phi$
for $i \neq j$

$\underline{\mathcal{F} \subseteq \mathcal{P}(X)}$ $\underline{\text{algebra}}$

(i) $\phi, X \in \mathcal{F}$. ✓

(ii) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$ ✓

(iii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$ ✓

$\underline{\mathcal{P}(X)}$ is algebra

$\underline{\mathcal{C} \subseteq \mathcal{P}(X)}$ ✓

$$\boxed{\mathcal{A} = \bigcap_{\mathcal{F} \in I} \mathcal{F}}$$

(i) $\phi \in \mathcal{A}$ $\quad$ ($\because$ $\phi \in \mathcal{F}$

$X \in \mathcal{A}$ $\quad$ (X '')

(ii) $\underline{E \in \mathcal{A}} \Rightarrow \underline{E \in \mathcal{F}} \ \forall \mathcal{F} \in I$

$\Rightarrow E^c \in \mathcal{F} \ \forall \mathcal{F} \in I$

$$\Rightarrow \underline{E^c} \in \bigcap_{\mathcal{F} \in I} \mathcal{F} \underline{= \mathcal{A}}$$

(iii) $E, F \in \mathcal{A} \Rightarrow E, F \in \mathcal{F}, \ \mathcal{F} \in I$

$\Rightarrow E \cap F \in \mathcal{F}, \ \mathcal{F} \in I$

$\Rightarrow E \cap F \in \mathcal{A}$

$$\boxed{\mathcal{A} = \bigcap_{\mathcal{F} \in I} \mathcal{F}}$$ is an algebra of subsets of X.

Also $\mathcal{C} \subseteq \mathcal{A}$.

Obvious $\mathcal{A}$ is the smallest algebra of subsets of X such that $\mathcal{C} \subseteq \mathcal{A}$.:

If $\mathcal{F}$ is any algebra, $\mathcal{C} \subseteq \mathcal{F}$

$\Longrightarrow \quad \mathcal{F} \supseteq \mathcal{A}$.

$$\underline{\underline{\mathcal{I}}} = \{ \mathcal{F} \mid \underline{\mathcal{F} \subseteq \mathcal{P}(X)}, \mathcal{F} \text{ is an } \underline{\text{algebra}}, \underline{\underline{\mathcal{C} \subseteq \mathcal{F}}} \}$$

$\underline{\mathcal{P}(X)} \in \mathcal{I}$

$\Longrightarrow \quad \mathcal{I} \neq \phi$

$$\mathcal{A} = \bigcap_{\mathcal{F} \in \mathcal{I}} \mathcal{F}$$

<u>Claim</u> (i) $\mathcal{C} \subseteq \mathcal{A}$

(ii) $\mathcal{A}$ is an algebra

(iii) $E, F \in \mathcal{F}(\mathcal{C}) \Longrightarrow \underline{E \cap F} \in \mathcal{F}(\mathcal{C})$?

$\Downarrow$ $\qquad$ $\underline{E \cup F} \in \mathcal{F}(\mathcal{C})$

E or E^c finite

F or F^c finite

<u>Case I</u> Both E, F are finite

$\Longrightarrow E \cup F$ is finite

$\Longrightarrow E \cup F \in \mathcal{F}(\mathcal{C})$

<u>Case II</u> Either E or F is not finite

Suppose E is not finite

$E \in \mathcal{F}(\mathcal{C}) \Longrightarrow E^c$ is finite

$$E \subseteq E \cup F$$

$$\Rightarrow (E \cup F)^c \subset E^c$$

$$\Rightarrow (E \cup F)^c \text{ is finite}$$

$$\Rightarrow (E \cup F) \in \mathcal{F}(\mathcal{C}).$$

<u>Hence</u>

$$E, F \in \mathcal{F}(\mathcal{C})$$

$$\Rightarrow E \cup F \in \mathcal{F}(\mathcal{C}).$$

$\mathcal{F}(\mathcal{C})$ <u>is an algebra</u>

$X =$ Any set ✓

$$\mathcal{C} = \{ \{x\} \mid x \in X \}$$

claim

$$\mathcal{F}(\mathcal{C}) = \{ A \in X \mid \underline{A} \text{ or } \underline{A^c} \text{ is finite} \}$$

(i) $\phi \in \mathcal{F}(\mathcal{C}), \; X \in \mathcal{F}(\mathcal{C})$ ✓

(ii) $\underline{E \in \mathcal{F}(\mathcal{C})} \Longrightarrow E^c \in \mathcal{F}(\mathcal{C})$

$\Longleftrightarrow E^c$ or $(E^c)^c$ finite

$\Longleftrightarrow E^c$ or E finite

$\Longleftrightarrow E^c \in \mathcal{F}(\mathcal{C})$

(a) $\underline{A \in \mathcal{F}(\mathcal{C})} \Longrightarrow A \underline{\in \mathcal{A}}$

A is finite:

$A = \{x_1, x_2, \dots, x_n\}$

$= \bigcup_{i=1}^{n} \underline{\{x_i\}}$

$\Longrightarrow A \in \cancel{\mathcal{C}\ \mathcal{F}(\mathcal{C})} \ \cancel{(\because \{x_i\} \in \mathcal{C} \subseteq \mathcal{F}(\mathcal{C}))}$

$\{x_i\} \in \mathcal{C} \subseteq \mathcal{A} \Longrightarrow \{x_i\} \in \mathcal{A}$

$\Longrightarrow A \in \mathcal{A}.$

$\mathcal{F}(\mathcal{C}) = \{A \subseteq X \mid A \text{ or } A^c \text{ finite}\}$

is an <u>algebra</u> and $\underline{\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})}$

$\forall\ x \in X$, $\{x\}$ is finite

$\Longrightarrow \{x\} \in \mathcal{F}(\mathcal{C})$

$\Longrightarrow \mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$

<u>Claim</u> $\mathcal{F}(\mathcal{C})$ is smallest, i.e.

let $\mathcal{A}$ be any algebra such

that $\underline{\mathcal{C} \subseteq \mathcal{A}}$.

$\overset{?}{\Longrightarrow} \underline{\mathcal{A} \supseteq \mathcal{F}(\mathcal{C})}$.

$$\mathcal{C} = \{\{x\} \mid x \in X\}$$

$$\mathcal{F}(\mathcal{C}) = \{A \subseteq X \mid A \text{ or } A^c \text{ is finite}\}$$

$$\|$$

The algebra generated by $\mathcal{C}$.

Given $\mathcal{C}$ a semi-algebra

$\mathcal{F}(\mathcal{C})$ - the algebra generated is

$$= \left\{ E \subseteq X \;\middle|\; E = \bigsqcup_{i=1}^{k} C_i, \; C_i \in \mathcal{C} \right\}$$

(i) $\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$, $\Rightarrow \phi, X \in \mathcal{F}(\mathcal{C})$.

(ii) $E \in \mathcal{F}(\mathcal{C}) \Rightarrow \underbrace{E = \bigsqcup_{i=1}^{n} C_i}$, $C_i \in \mathcal{C}$

Recall $C_i \in \mathcal{C}$, a semi-algebra

$$\Longrightarrow C_i^c = \bigsqcup_{j=1}^{k_i} A_j^i, \quad A_j^i \in \mathcal{C}$$

$$\Longrightarrow E^c = \bigcap_{i=1}^{n} C_i^c = \underbrace{\bigcap_{i=1}^{n} \left[\bigsqcup_{j=1}^{k_i} A_j^i \right]}$$

$$= \bigcup \underbrace{(A_j^i \cap A_l^k)}_{\in \mathcal{C}} \qquad \boxed{\begin{array}{l} 1 \le i \le n_1 \\ 1 \le k \le n_2 \\ 1 \le j, l \le l \end{array}}$$

$$E \in \mathcal{F}(\mathcal{C}) \implies E^c \in \mathcal{F}(\mathcal{C})$$

(iii) $E, F \in \mathcal{F}(\mathcal{C}) \implies E \cap F \in \mathcal{F}(\mathcal{C})$

$$E = \bigcup_{i=1}^{n} A_i, \quad A_i \in \mathcal{C}$$

$$F = \bigcup_{j=1}^{m} B_j, \quad B_j \in \mathcal{C}$$

$$E \cap F = \left(\bigsqcup_{i=1}^{n} A_i\right) \cap \left(\bigsqcup_{j=1}^{m} B_j\right)$$

$$\cancel{\bigsqcup_{i,j} \underbrace{(A_i \cap B_j)}_{\in \mathcal{C}}}$$

$$= \bigsqcup_{i,j} (\underbrace{A_i \cap B_j}_{\in \mathcal{C}})$$

$$\Rightarrow E \cap F \in \mathcal{F}(\mathcal{C})$$

If $\mathcal{A}$ is an algebra and $\mathcal{A} \supseteq \mathcal{C}$

$\Longrightarrow \mathcal{A} \supseteq \mathcal{F}(\mathcal{C})$

Let $E \in \mathcal{F}(\mathcal{C})$

$\Longrightarrow E = \bigsqcup_{i=1}^{n} A_i$, $A_i \in \mathcal{C} \subseteq \mathcal{A}$

$\Longrightarrow E \in \mathcal{A}$

Hence $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{A}$.

$\mathcal{C}$ — Any collection of subsets of X

$E \subseteq X$ fixed.

Define $\mathcal{C} \cap E := \{ C \cap E \mid C \in \mathcal{C} \}$

A collection of subsets of the set E

$\mathcal{F}(\mathcal{C} \cap E)$ — Algebra of subsets of E generated by $\mathcal{C} \cap E$

Claim $\boxed{\mathcal{F}(\mathcal{C} \cap E) = \mathcal{F}(\mathcal{C}) \cap E}$

Note: $\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$ ✓

$$\Rightarrow \boxed{\mathcal{C} \cap E \subseteq \mathcal{F}(\mathcal{C}) \cap E}$$ ✓

$\left(\because A \in \mathcal{C} \cap E \Rightarrow A \cap E \in \mathcal{C} \subseteq \mathcal{F}(\mathcal{C}) \right)$

$A \in \mathcal{C})$ $\Downarrow$

Observe $\mathcal{F}(\mathcal{C}) \cap E)$ is an algebra of subsets of E?

(i) $\phi = \phi \cap E \in \mathcal{F}(\mathcal{C}) \cap E)$

$E = X \cap E \in \mathcal{F}(\mathcal{C}) \cap E$.

$\Longrightarrow \quad \mathcal{F}(\mathcal{C} \cap E) \subseteq \mathcal{F}(\mathcal{C}) \cap E$

<u>Claim</u> $\quad \mathcal{F}(\mathcal{C}) \cap E \subseteq \mathcal{F}(\mathcal{C} \cap E)$?

Define
$$\boxed{\mathcal{A} = \{A \subseteq X \mid A \cap E \in \underline{\mathcal{F}(\mathcal{C} \cap E)}\}}$$

<u>Note</u> $\quad \underline{\mathcal{C} \subseteq \mathcal{A}} \quad \left(\because A \in \mathcal{C} \implies A \cap E \in \mathcal{C} \cap E \subseteq \mathcal{F}(\mathcal{C} \cap E) \right.$

<u>If</u> $\mathcal{A}$ <u>is an algebra</u>

$\Longrightarrow \quad \underline{\mathcal{F}(\mathcal{C}) \subseteq \mathcal{A}}$.

$F \in \mathcal{F}(\mathcal{C}), \quad F \cap E \in \mathcal{F}(\mathcal{C} \cap E)$

$\mathcal{F}(\mathcal{C}) \cap E \;\underline{\subseteq \mathcal{F}(\mathcal{C} \cap E)}$

(i) $\phi \in \mathcal{A}$ $(\because \phi \cap E = \phi \in \mathcal{F}(\mathcal{C} \cap E)$

$X \in \mathcal{A}$ $(\because X \cap E = E \in \underline{\mathcal{F}(\mathcal{C} \cap E)}$

(ii) $\underline{G \in \mathcal{A}} \Rightarrow G \cap E \in \mathcal{F}(\mathcal{C} \cap E)$

$\downarrow$

Algebra

$\Rightarrow G^c \cap E \in \mathcal{F}(\mathcal{C} \cap E)$

$\Rightarrow \underline{G^c \in \mathcal{A}}$

(iii) $\underline{G, H \in \mathcal{A}} \Rightarrow G \cap E, H \cap E \in \mathcal{F}(\mathcal{C} \cap E)$

$\Rightarrow \underline{(G \cap H) \cap E \in \mathcal{F}(\mathcal{C} \cap E)}$

(ii) $A, B \in \mathcal{F}(\mathcal{C}) \cap E$

$\Rightarrow A = G \cap E, \quad G \in \mathcal{F}(\mathcal{C})$

$B = H \cap E, \quad H \in \mathcal{F}(\mathcal{C})$

$\Rightarrow A \cap B = (\underbrace{G \cap H}_{\in \mathcal{F}(\mathcal{C})}) \cap E.$

$\Rightarrow A \cap B \in \mathcal{F}(\mathcal{C}) \cap E$

(iii) $A \in \mathcal{F}(\mathcal{C}) \cap E$

$\Rightarrow A = G \cap E, \quad G \in \mathcal{F}(\mathcal{C})$

$\underline{A^c} = G^c \cap E \in \mathcal{F}(\mathcal{C}) \cap E$