Measure and Integration

Lecture - 3

10/11/10

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$$\mathcal{A} \subseteq \mathcal{P}(X)$$

Algebra

$$\begin{cases} \text{(i)} & \emptyset, X \in \mathcal{A} \\ \text{(ii)} & A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{A} \\ \text{(iii)} & A, B \in \mathcal{A} \Longrightarrow A \cap B \in \mathcal{A} \end{cases}$$

$$\equiv A, B \in \mathcal{A} \Longrightarrow A \cup B \in \mathcal{A}$$

$\mathcal{C}$ need not be an algebra

$\mathcal{F}(\mathcal{C})$ – smallest algebra including $\mathcal{C}$.

$$\mathcal{C} \cap E = \{ A \cap E \mid A \in \mathcal{C} \}$$

$$\mathcal{F}(\mathcal{C} \cap E) = \mathcal{F}(\mathcal{C}) \cap E$$

Suppose $\mathcal{A}$ is an algebra

$A_1, A_2, \ldots, A_n \ldots \in \mathcal{A}$

$$E = \bigcup_{n=1}^{\infty} A_n$$

Define

$$B_1 := A_1$$

$$B_2 := A_2 \setminus A_1$$

$$B_3 := (A_1 \cup A_2) \setminus A_3$$

$$B_n := \left(\bigcup_{i=1}^{n-1} A_i \right) \setminus A_n$$

$$B_n = \underline{\left(\bigcup_{i=1}^{n-1} A_i\right)} \cap \underline{A_n^c}\ .$$

$$\Longrightarrow \quad B_n \in \mathcal{A} \ \forall n.$$

$$B_n \cap B_m = \phi \text{ for } n \neq m$$

<u>Further</u>

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

$$\Longrightarrow \quad \bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} A_i = E$$

X — uncountable

$\mathcal{F} = \{E \subseteq X \mid E \text{ or } E^c \text{ finite}\}$

Then $\mathcal{F}$ is an algebra. ✓

<u>Q</u>: Is $\mathcal{F}$ a σ-algebra?

i.e., $E_1, E_2, \ldots, E_n, \ldots \in \mathcal{F}$

$$\Longrightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}\,?$$

<u>Note</u> <u>X is uncountable</u>.

$\exists$ a subset $E \subseteq X$ such that E is countably infinite and E^c is not finite, i.e. E^c is infinite

If not

$$X = E \cup E^c$$

E: countable, E^c: finite

$\Rightarrow X$ is countable. $\otimes$

$\underline{E} = \{x_1, x_2, \ldots, x_n, \ldots\}$

$= \bigcup_{i=1}^{\infty} \{x_i\}$ ✓

<u>N.B</u> $\{x_i\} \in \mathcal{F}$.

$\not\Rightarrow E \notin \mathcal{F}$.

$\mathcal{S} = \{A \subseteq X \mid A$ or A^c is countable$\}$

(i) $\phi \in \mathcal{S}$, $X \in \mathcal{S}$.

(ii) $A \in \mathcal{S} \Leftrightarrow A^c \in \mathcal{S}$

(iii) $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$

$$\Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{S} \ ?$$

Case (i) All A_n's are countable

$$\Longrightarrow \underline{\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}}$$

Case (ii) $\exists\, n_0$ s.t. $A_{n_0} \in \mathcal{S}$

but A_{n_0} is not countable

$\Longrightarrow$ $A_{n_0}^c$ is countable

Observe $\bigcup_{n=1}^{\infty} A_n \supseteq A_{n_0}$

$\Longrightarrow$ $\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq A_{n_0}^c \to$ countable

$\Longrightarrow$ $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$ is countable

$\Longrightarrow\ \in \mathcal{S}$

$$\underline{\mathcal{S}(\mathcal{C})} = \bigcap_{\substack{\mathcal{S}\ \text{algebras} \\ \mathcal{S} \supseteq \mathcal{C}}} \mathcal{S}$$

σ-algebras

(i) $\quad \phi, X \in \mathcal{S}(\mathcal{C})$.

(ii) $\quad A \in \mathcal{S}(\mathcal{C}) \Rightarrow A \in \mathcal{S} \ \forall \mathcal{S}$

$\Rightarrow A^c \in \mathcal{S} \ \forall \mathcal{S}$

$\Rightarrow A^c \in \bigcap \mathcal{S} = \mathcal{S}(\mathcal{C})$

$\vdots$

(iii) $A_n \in \mathcal{S}(\mathcal{C}) \ \forall n$

$$\Rightarrow A_n \in \mathcal{S} \ \forall \ \underline{\mathcal{S}}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{S} \ \forall \mathcal{S}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \bigcap_{\mathcal{S}} \mathcal{S} = \mathcal{S}(\mathcal{C})$$

$\mathcal{S}(\mathcal{C})$ is a σ-algebra.

$\mathcal{C} \subseteq \underline{\mathcal{S}(\mathcal{C})}$ ✓

X any set

$$\mathcal{S}(\mathcal{E}) = \{ A \subseteq X \mid A \text{ or } A^c \text{ is countable} \}$$

$$\| $$

$$\mathcal{S}$$

?

(1) $\mathcal{S}$ is a σ-algebra.

(2) $\mathcal{E} \subseteq \mathcal{S}$, $(\{x\} \in \mathcal{S}.)$

(3) $\mathcal{S}$ is smallest ?

Let $\mathcal{F}$ be any σ-algebra such that $\underline{\mathcal{E} \subseteq \mathcal{F}}$

To show

$\mathcal{F} \supseteq \mathcal{S}$?

let $A \in \mathcal{S}$

either

A is countable, | A^c is countable

$A = \{x_1, x_2, \ldots\}$

$= \bigcup_{i=1}^{\infty} \{x_i\} \in \mathcal{F}$

$\Downarrow$

$A^c \in \mathcal{F}$

$\Downarrow$

$A \in \mathcal{F}$.

$(X, \mathcal{F})$

$\mathcal{F}$ - Topology

(Open sets in X)

$\phi, X \in \mathcal{F}$

$E_1, E_2 \in \mathcal{F} \Rightarrow E_1 \cap E_2 \in \mathcal{F}$

$E_\alpha \in \mathcal{F} \Rightarrow \bigcup_\alpha E_\alpha \in \mathcal{F}$

$\mathcal{F}$ is a topology, it need not be an algebra σ-algebra!

$\mathcal{U} =$ open sets
$\mathcal{C} =$ closed sets

Claim

$$S(\mathcal{U}) = S(\mathcal{C})$$

let $\underline{E} \in \underline{\mathcal{U}} \iff E$ is open

$\iff E^c$ is closed

$\iff \underline{E^c \in \mathcal{C}} \subseteq \underline{S(\mathcal{C})}$

$\iff E \in S(\mathcal{C})$

$\implies \underline{\mathcal{U}} \subseteq \underline{S(\mathcal{C})}$

$$\Rightarrow \quad \underline{\mathcal{S}(\mathcal{U}) \subseteq \mathcal{S}(\mathcal{C})}$$

$$\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}(\mathcal{U})$$

$$A \in \mathcal{C} \Rightarrow A^c \in \mathcal{U} \subseteq \underline{\mathcal{S}(\mathcal{U})}$$

$$\Rightarrow A \in \mathcal{S}(\mathcal{U})$$

$$\Rightarrow \mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}(\mathcal{U}$$

$\phi \in \underline{S}(\mathcal{E}) \cap Y \quad (\because \phi = \phi \cap Y$

$Y \in \underline{S}(\mathcal{E}) \cap Y \quad (Y = \underline{X} \cap Y$

(ii) $E \in \underline{S(\mathcal{E}) \cap Y}$; $E = A \cap Y$

$E^c (\text{in } Y)$ $\quad (A \in \underline{S}($

$= E^c \cap Y = \underline{A^c \cap Y}$

$\in \underline{S}(\mathcal{E}) \cap Y$

(iii) $E_n \in \underline{S(\mathcal{E}) \cap Y}$ $\quad E = A_n \cap Y,\ \underline{A_n \in S(\mathcal{E}}$

$\Longrightarrow \bigcup E_n = (\bigcup A_n) \cap Y$

Claim

$$\| \mathcal{S}(\mathcal{E}) \cap Y \subseteq \underline{\mathcal{S}(\mathcal{E} \cap Y)}$$

Let

$$\mathcal{A} := \{ E \subseteq X \mid E \cap Y \in \mathcal{S}(\mathcal{E} \cap Y) \}$$

Shows

$$\| \mathcal{A} \text{ is a } \sigma\text{-algebra} \|$$

$$?\, \| \underline{\mathcal{E} \subseteq \mathcal{A}}$$

$$\overset{!}{\Longrightarrow} \underline{\mathcal{S}(\mathcal{E}) \subseteq \mathcal{A}}$$

(i) $\phi \in \mathcal{A}$ ✓

$X \in \mathcal{A}$ ✓

(ii) $E \in \mathcal{A} \Rightarrow E \cap Y \in \underline{S(Y}$…

$\Rightarrow E^c \cap Y \in S(Y$…

$\Rightarrow E^c \in \mathcal{A}.$

(iii) $E_n \in \mathcal{A} \Rightarrow E_n \cap Y \in S(Y$…

$\Rightarrow (\underline{\cup E_n}) \cap Y \in S(Y$…

$\Rightarrow \cup E_n \in \mathcal{A}.$

$\mathcal{C} \subseteq \mathcal{P}(X)$, $Y \subseteq X$

$\underline{S(\mathcal{C} \cap Y) = S(\mathcal{C}) \cap Y}$?

Note

$$\mathcal{C} \subseteq S(\mathcal{C})$$

$$\mathcal{C} \cap Y \subseteq \boxed{S(\mathcal{C}) \cap Y}$$

is an σ-algebra

$$\Longrightarrow \underline{S(\mathcal{C} \cap Y) \subseteq S(\mathcal{C}) \cap Y}.$$

$\mathcal{C} \subseteq \mathcal{P}(X)$

$\therefore$

$\underline{\mathcal{S}(\underline{\mathcal{A}(\mathcal{C})}) = \mathcal{S}(\mathcal{C})}$

<u>Note</u> $\underline{\mathcal{C}} \subseteq \mathcal{A}(\mathcal{C}) \subseteq \underline{\mathcal{S}(\mathcal{A}(\mathcal{C}))}$

$\Rightarrow \underline{\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}(\mathcal{A}(\mathcal{C}))}$

<u>Also</u> $\mathcal{C} \subseteq \underline{\mathcal{S}(\mathcal{C})}$ ✓

$\Rightarrow \underline{\mathcal{A}(\mathcal{C})} \subseteq \underline{\mathcal{S}(\mathcal{C})}$

$\Rightarrow \mathcal{S}(\mathcal{A}(\mathcal{C})) \subseteq \mathcal{S}(\mathcal{C})$ ✓