Measure Theory

Lecture 5

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$$\mu(A) = \sum_{\{i : x_i \in A\}} p_i$$

μ is countably additive. Let

$$A = \bigcup_{i=1}^{n} A_i, \quad A_i \subseteq X, \quad A_i \cap A_j = \phi$$

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i).$$

Enough to check when

$A_i = \{x_{k_i}\}$ $i \geq 1$

Then $\mu(A) = \underline{\sum_{i=1}^{\infty} p_{k_i}} = \lim_{n \to \infty} \left(\sum_{i=1}^{n} p_{k_i} \right)$

$$\mu(A) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} p_{k_i}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \mu(A_i)\right)$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

A~~x~~

$$\mu(A) = \cancel{p} \sum_{i=1}^{\infty} \mu(A_i)$$

$$A_i \subseteq X,\ A_i \cap A_j = \phi.$$

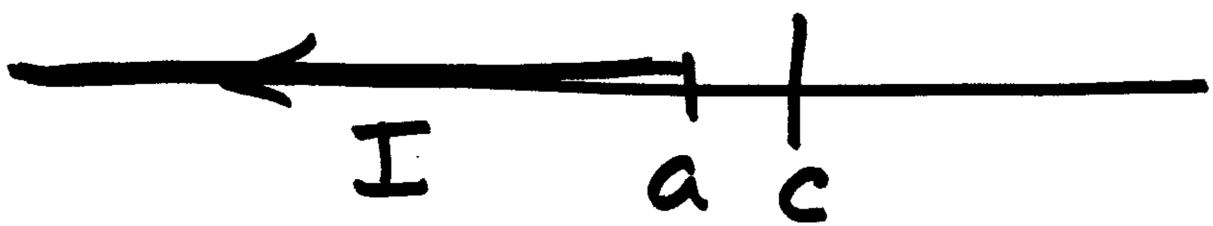
$I, J \in \mathcal{I}$, $I \subseteq J$

$\Longrightarrow \lambda(I) \leq \lambda(J)$

<u>Case I</u> $\quad I$ is infinite, say

let $\quad \begin{array}{l} I = (-\infty, a] \\ J = (-\infty, c] \end{array}$

where $\quad c \geq a$



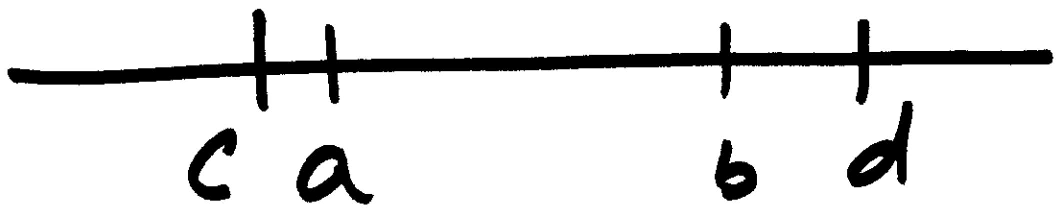
$\lambda(I) = +\infty = \lambda(J)$

$\underline{I} \subseteq J$, J is infinite

$$\lambda(I) \leq +\underline{\infty = \lambda(J)}$$

<u>Case</u> both I and J are finite

$I \subseteq J$



$$I(a,b) \subseteq J(c,d)$$

$$\Rightarrow \quad c \leq a \leq b \leq d.$$

i.e. $\quad d-c \geqslant b-a$

$$\lambda(J) \geqslant \lambda(I).$$

(5)

$$I = \bigcup_{i=1}^{n} J_i \quad , \quad J_i \cap J_k = \phi$$

$$\Longrightarrow \lambda(I) = \sum_{i=1}^{n} \lambda(J_i)$$

If I is infinite, $I = \bigcup_{i=1}^{n} J_i$ ✓

$\Longrightarrow$ At least one of J_i is infinite

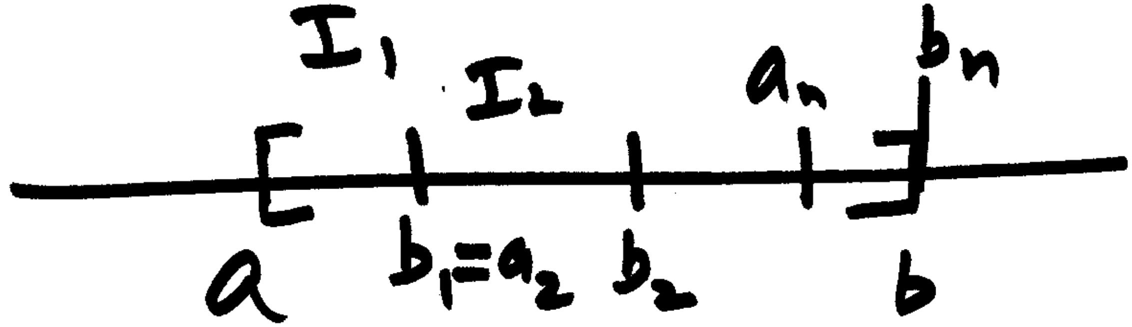
$$\Longrightarrow \quad \lambda(I) = +\infty = \sum_{i=1}^{n} \lambda(J_i)$$

I finite

$$I = \bigcup_{i=1}^{n} J_i \, , \quad J_i \cap J_k = \phi, \; i \neq k$$

$I = I(a,b)$. Note $\lambda(I) = \lambda([a,b])$

w.l.g., let $I = [a,b]$



We can assume $I_1 = [a, b_1]$

- - - - - Arrange the end points of $J_1, J_2, \ldots, J_n$ such that

$$a = a_1 \le b_1 = a_2 \le b_2 \cdots < a_n \le b_n = b$$

$$b - a = b_n - a_1$$
$$= \sum_{i=1}^{n} (b_i - a_i)$$

$$\lambda(I) = \sum_{i=1}^{n} \lambda(J_i)$$

$$I \subseteq \bigcup_{i=1}^{n} I_i$$

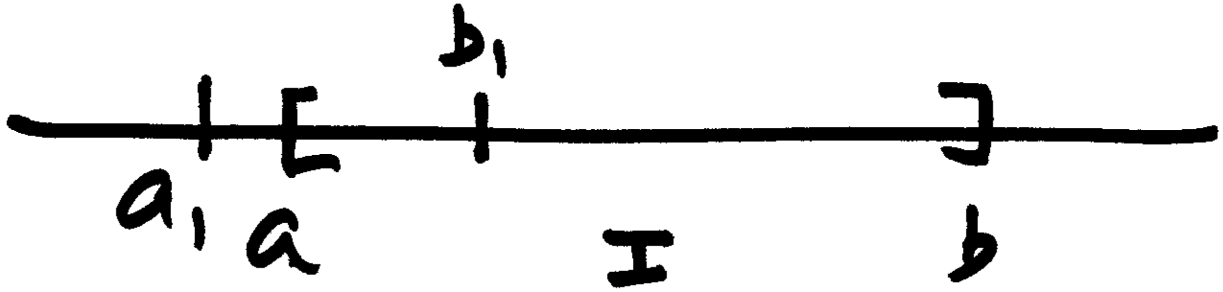
If One of I_i is infinite

$$\Longrightarrow \quad \lambda(I) \leq \sum_{i=1}^{n} \lambda(I_i)$$

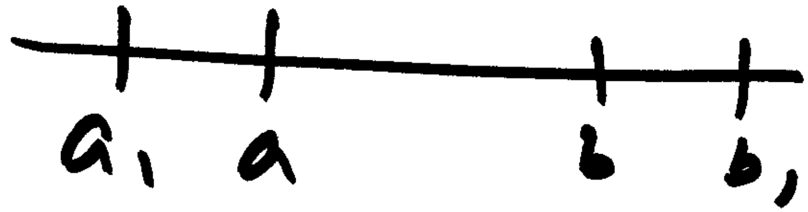
<u>Suppose</u> each I_i is finite

$\Longrightarrow$ I is finite.

<u>Assume(!)</u> $I = [a, b]$



Either $b_1 \geq b$

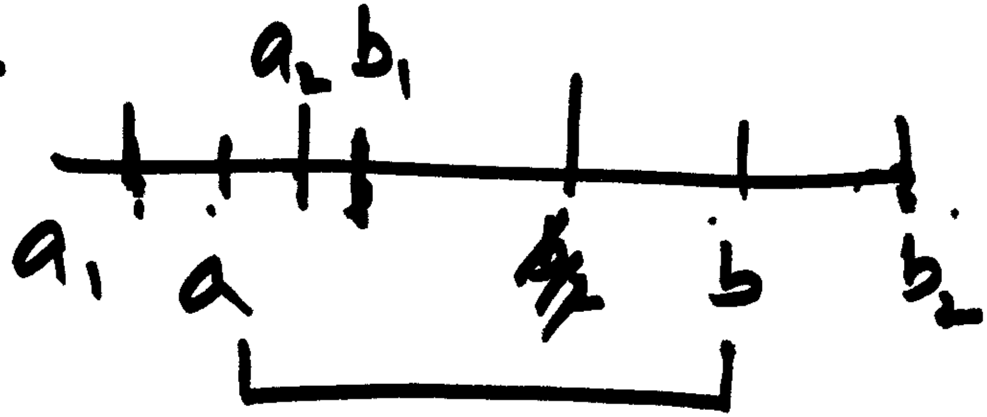


Then $\lambda(I) = b - a \leq b_1 - a_1$

$$= \lambda(I_1)$$

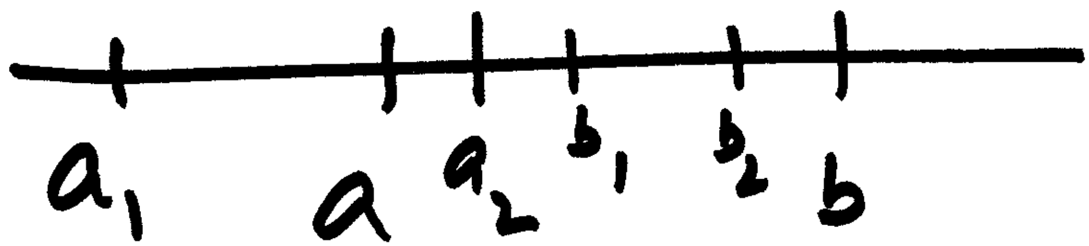
$$\leq \sum_{i=1}^{n} \lambda(I_i)$$

<u>Suppose</u>



$$\lambda(I) = b - a \leq b_2 - a_1 \leq b_2 - a_2 + b_1 - a_1$$

$$\lambda(I) \leq \lambda(I_1)+\lambda(I_2)$$
$$\leq \sum_{i=1}^{n} \lambda(I_i)$$



or

$$a_1 \leq a \leq a_2 \leq b_1 \leq b_2 \leq \cdots \leq a_n \leq b \leq b_n$$

$$\lambda(I) = b-a \leq b_n - a_1$$
$$\left.\begin{aligned} \leq\ & b_n - a_n \\ +\ & b_{n-1} - a_{n-1} \\ & \cdots \\ +\ & b_1 - a_1 \end{aligned}\right\} = \sum_{j=1}^{n} \lambda(I_j)$$

$$I \subseteq \bigcup_{i=1}^{\infty} I_i, \quad I \text{ finite}$$

$$\Longrightarrow \lambda(I) \leqslant \sum_{i=1}^{\infty} \lambda(I_i) \quad !$$

<u>Note</u> if I_i is infinite for some i,

then $\lambda(I_i) = +\infty$

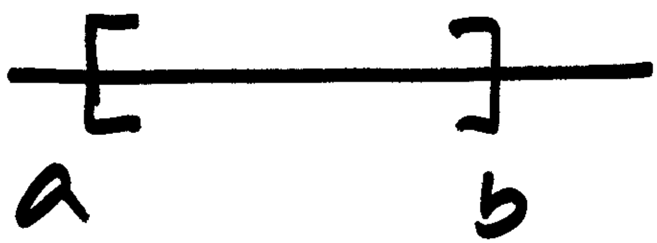
$\geq \lambda(I)$

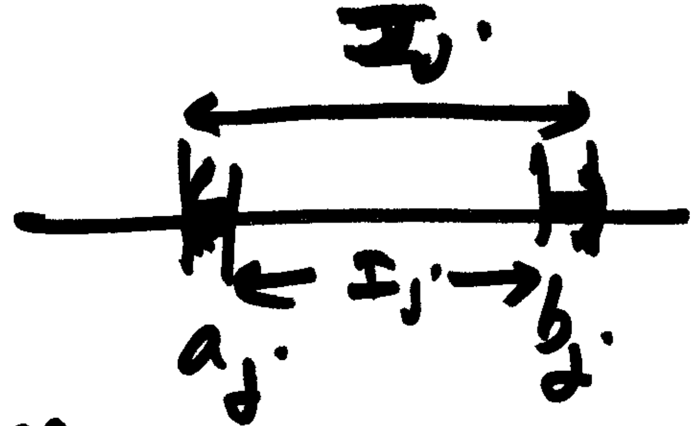
$$\Rightarrow \sum_{j \geq 1}^{\infty} \lambda(I_j) \geq \lambda(I)$$

✓.

I finite, say $[a,b]$

Each I_j is finite with left end point a_j, right end point b_j





$\varepsilon/2^j$

$$I = [a,b] \subseteq \bigcup_{j=1}^{\infty} I_j$$

Let $\varepsilon > 0$ be fixed. Select an open interval J_j such that

$$J_j \supseteq I_j \text{ and } \lambda(J_j) = \lambda(I_j) + \varepsilon/2^j$$

$$[a,b] \subseteq \bigcup_{j=1}^{\infty} I_j \subseteq \bigcup_{j=1}^{\infty} J_j$$ (13)

$$\xrightarrow[\text{Property}]{\text{Hine Borel}} \exists \; n \text{ s.t}$$

$$I = [a,b] \subseteq \bigcup_{j=1}^{n} J_j$$

$$\Longrightarrow \quad \lambda(I) \leq \sum_{j=1}^{n} \lambda(J_j)$$

$$\leq \sum_{j=1}^{n} \left[\lambda(I_j) + \frac{\varepsilon}{2^j}\right]$$

$$\leq \sum_{j=1}^{\infty} \lambda(I_j) + \underbrace{\sum_{j=1}^{\infty} \varepsilon/2^j}_{= \varepsilon}$$

let $\varepsilon \to 0$

(14)

$$I = \bigcup_{j=1}^{\infty} I_j \, , \quad I_j \cap I_k = \phi$$

I finite

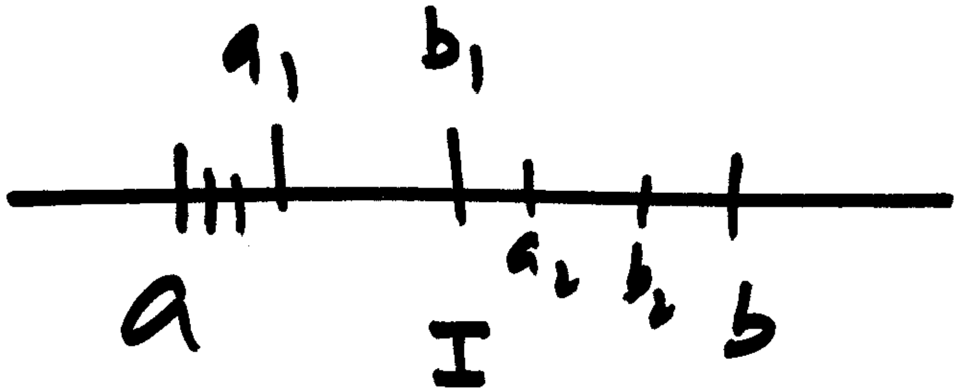
$$\Longrightarrow \lambda(I) = \sum_{j=1}^{\infty} \lambda(I_j) .$$

Note

$$\lambda(I) \leq \sum_{j=1}^{\infty} \lambda(I_j) . \qquad (1)$$

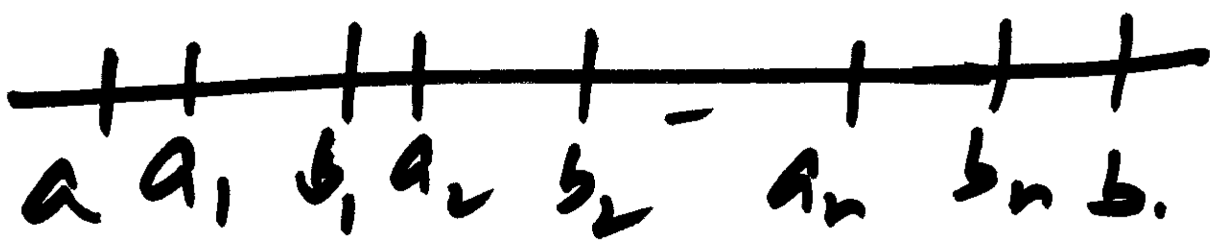
Only to show

$$\lambda(I) \geq \sum_{j=1}^{\infty} \lambda(I_j) \, ?$$



$I_1 \subseteq I$, $I_2 \subseteq I$

For any n, let consider the end points a_n, b_n of I_n's. We can arrange them such that



$$a \le a_1 \le b_1 \le a_2 \le b_2 \cdots \le a_n \le b_n \le b$$

$$\Rightarrow \lambda(I) = b - a$$

$$\ge b_n - a_1$$

$$\ge \left.\begin{aligned} & b_n - a_n \\ & + b_{n-1} - a_{n-1} \\ & - \_\_\_\_\_\_ \end{aligned}\right\}$$

$$+ b_1 - a_1$$

$$= \sum_{i=1}^{n} \lambda(I_i) \quad \forall n$$

$$\Rightarrow \lambda(I) \ge \sum_{i=1}^{\infty} \lambda(I_i).$$