Lecture 6

Measure & Integration

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Finite

$$I = \bigcup_{n=1}^{\infty} I_n , \quad I_n \cap I_m = \phi$$

$$\Rightarrow \lambda(I) = \sum_{n=1}^{\infty} \lambda(I_n)$$

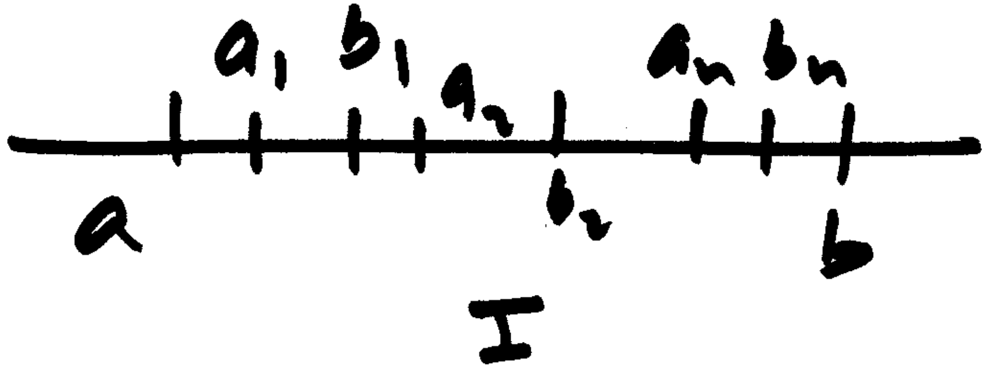
Recall $\lambda(I) \le \sum_{n=1}^{\infty} \lambda(I_n)$

To show $\lambda(I) \ge \sum_{i=1}^{\infty} \lambda(I_i)$?

Note for any n

$$I_1, I_2, \dots, I_n \subseteq I$$

Let



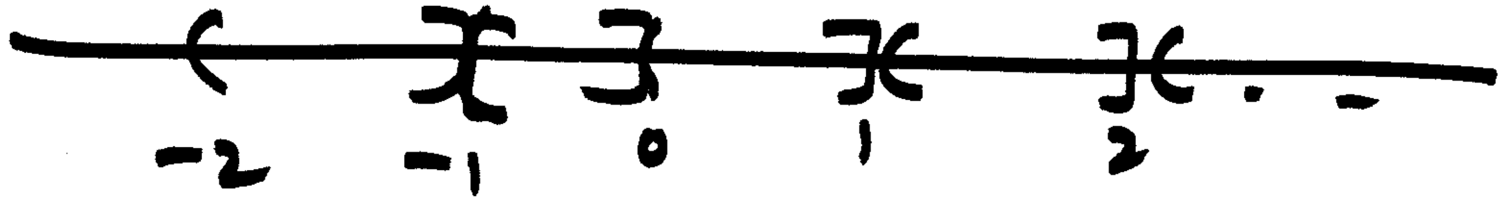
W.l.o.g. $a \le a_1 < b_1 \le a_2 \le b_2 \le \cdots < b_n \le b$

②

$$
\begin{aligned}
b - a &\geq b_n - a_1 \\
&\geq b_n - a_n \\
&\quad + a_{n-1} - a_{n-1} \\
&\quad \vdots \\
&\quad + b_1 - a_1
\end{aligned}
$$

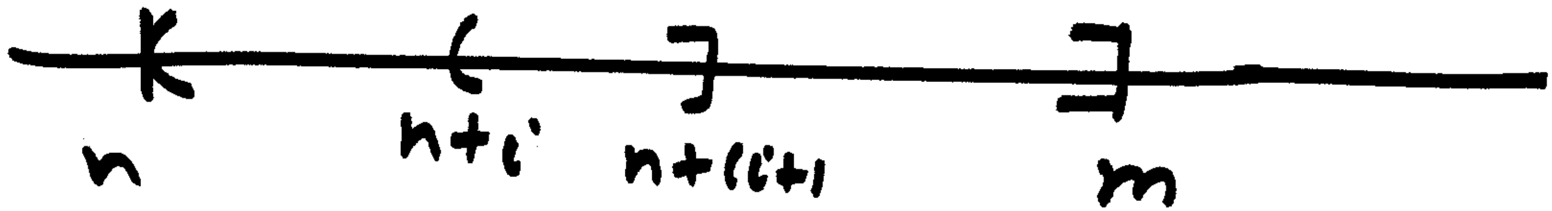
$$\lambda(I) \geq \sum_{i=1}^{n} \lambda(I_i) \quad \forall n$$

$$\Longrightarrow \lambda(I) \geq \sum_{i=1}^{\infty} \lambda(I_i)$$



If I is finite, then $\exists\, n, m$ such that

$$I \subset (n, m]$$



$$I = \bigcup_{i=1}^{i=m-n} \big((n, n+i) \cap I\big) \text{ (disjoint)}$$

$$\cancel{\Rightarrow \lambda(I) = \sum_{i=1}^{m-n} \lambda\big((n, n+i] \cap I\big) \quad \sum_{i=1}^{m-1}}$$

$\cancel{I = \bigcup (n, n+1}$ (4)

$$I = \bigcup_{i=0}^{m-n-1} \big((n+i, n+i+1] \cap I \big)$$

$$\Rightarrow \quad \lambda(I) = \sum_{i=0}^{m-n-1} \lambda\big((n+i, n+i+1] \cap I \big)$$

$$= \sum_{i \in \mathbb{Z}} \lambda\big((n+i, n+i+1] \cap I \big)$$

I finite ✓

I infinite, say $(a, +\infty)$

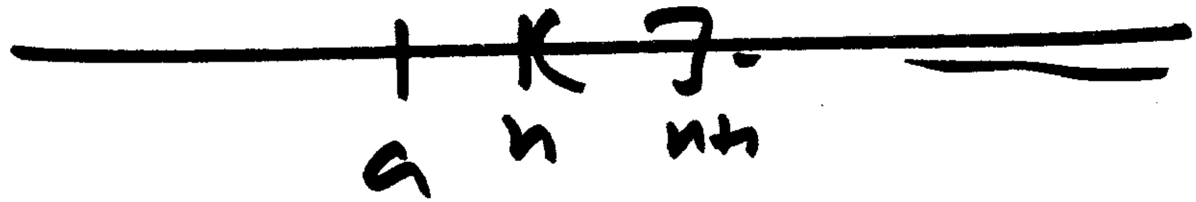
$$I = \bigcup_{i \in \mathbb{Z}} \left((n, n+1] \cap I \right)$$

$$\left(\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+1] \right)$$

I infinite

$\Longrightarrow$ $(n, n+1] \cap I = (n, n+1]$ for infinite n's

In fact if $n \geq a$, then



$$\Rightarrow \sum_{n \in \mathbb{Z}} \lambda\left((n, n+1] \cap I \right) = +\infty$$

$$= \lambda(I)$$

$$I = \bigcup_{n=1}^{\infty} I_n , \quad I_n \cap I_m = \phi$$

To show $\lambda(I) = \sum_{n=1}^{\infty} \lambda(I_n)$.

Proof

Note $\lambda(I) = \sum_{n \in \mathbb{Z}} \lambda(I \cap (n, n+1])$.

And $I \cap (n,n+1] = (n,n+1] \cap \left(\bigcup_{j=1}^{\infty} I_j \right)$

$$= \bigcup_{j=1}^{\infty} \left(I_j \cap (n,n+1] \right)$$

$$\Rightarrow \quad \lambda(I \cap (n,n+1]) = \sum_{j=1}^{\infty} \lambda(I_j \cap (n,n+1])$$

$$\lambda(I) = \sum_{n\in\mathbb{Z}} \left(\sum_{j=1}^{\infty} \lambda(I_j \cap (n, n+1]) \right)$$

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$$= \sum_{j=1}^{\infty} \left(\sum_{n\in\mathbb{Z}} \lambda(I_j \cap (n, n+1]) \right)$$

$$\lambda(I_j) = \sum_{n\in\mathbb{Z}} \lambda(I_j \cap (n, n+1])$$

$$\lambda(I) = \sum_{j=1}^{\infty} (\lambda(I_j))$$

$$I \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\begin{aligned}
\lambda(I) &= \sum_{n \in \mathbb{Z}} \lambda\big(\underline{I \cap (n, n+1]}\big) \\
&\leq \sum_{n \in \mathbb{Z}} \left(\sum_{j=1}^{\infty} \lambda\big(\underline{I_j \cap (n, n+1]}\big) \right) \\
&= \sum_{j=1}^{\infty} \left(\underline{\sum_{n \in \mathbb{Z}} \lambda\big(I_j \cap (n, n+1]\big)} \right) \\
&= \sum_{j=1}^{\infty} \lambda(I_j)
\end{aligned}$$

$$\underline{I} = I(a, b)$$

$$\underline{I + x} = I(a+x, b+x)$$

$$\lambda(I + x) = (b + x) - (a + x)$$

$$= b - a = \lambda(I)$$

$$I = (a, +\infty) \checkmark$$

$$I + x = (a + x, +\infty) \checkmark$$

$$\lambda(I) \underline{= +\infty =} \lambda(I + x)$$

$$I = \bigcup_{j=1}^{n} I_j \qquad I_j \cap I_k = \phi$$

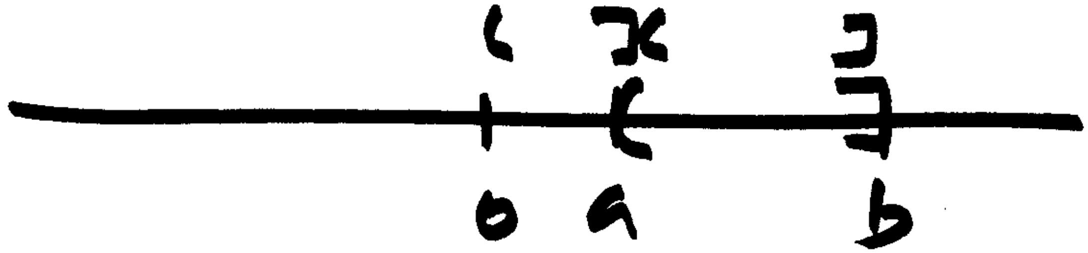
$$= \bigcup_{j=1}^{\infty} I_j, \qquad \underline{I_j = \phi} \ \forall\ j \geq n$$

$$\lambda(I) = \sum_{j=1}^{\infty} \lambda(I_j)$$

$$= \sum_{j=1}^{n} \lambda(I_j)$$

$(a, b]$ ✓

$$\mu(a, b] = F(b) - F(a).$$



$$F(b) - F(a) = \mu(0, b] - \mu(0, a]$$

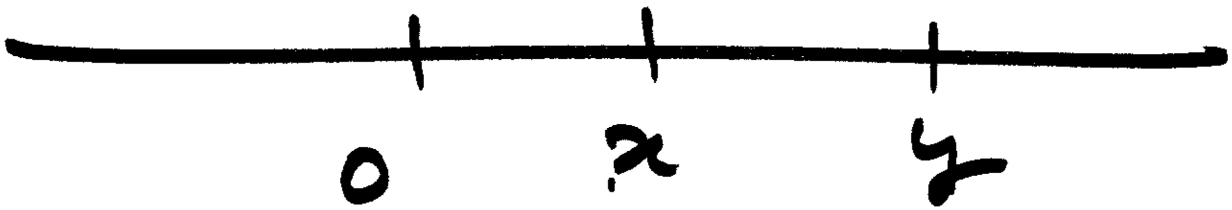
$$= \mu(a, b]$$ ✓

$\cancel{I = (a, +\infty) = }$

$\mu\cancel{(I) = }\ F$

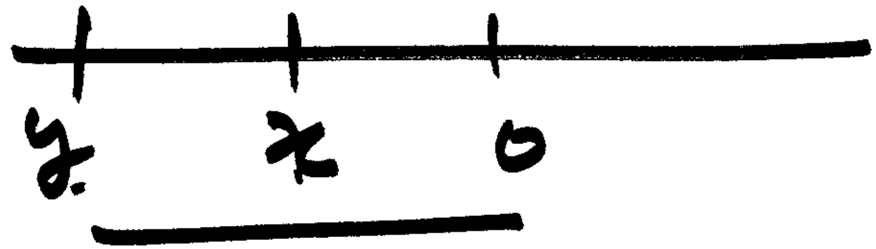
F is monotonically increasing?

$x < y$



$$
\begin{aligned}
F(y) &= \mu(0,y] \\
&= \mu\big((0,x] \cup (x,y]\big) \\
&= \mu(0,x] + \mu(x,y] \\
&= F(x) + \mu(x,y] \\
&\geq F(x)
\end{aligned}
$$

$$y \le x$$



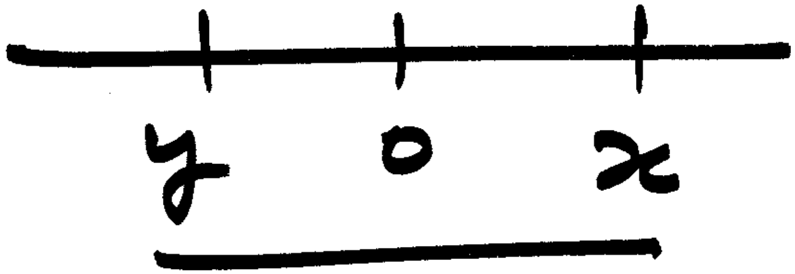
$$F(y) = -\mu(y, 0]$$

$$= -[\mu(y, x] \cup (x, 0])$$

$$= -\mu(y, x] - \mu(x, 0]$$

$$= -\mu(y, x) + F(x)$$

$$F(y) \le F(x)$$



$\cancel{F(y,x] =}$

$$0 \le \mu(y,x] = F(x) - F(y)$$

$$\Rightarrow \quad \underline{F(y) \le F(x)}$$

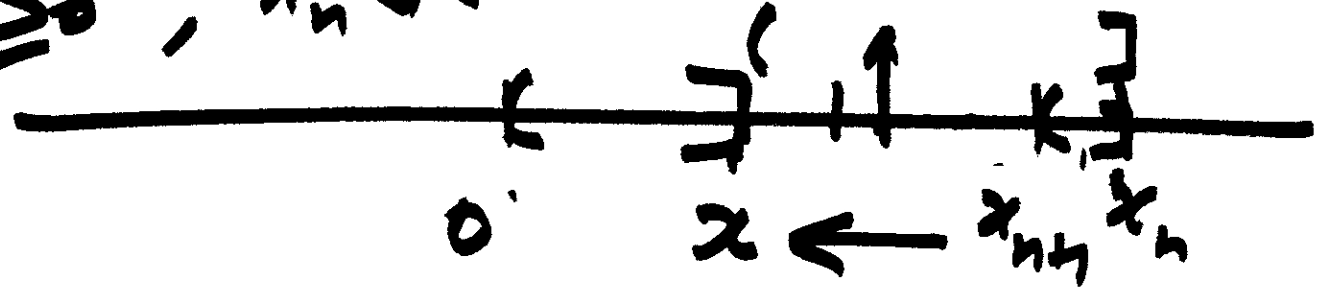
If μ is c.a

$\Longrightarrow$ F is right continuous.

Let $x \in \mathbb{R}$, $\{x_n\}_{n\geq 1}$ $x_n \downarrow x$

To show $F(x_n) \longrightarrow F(x)$

<u>Case</u> $x \geq 0$, $x_n \downarrow x$



<u>Note</u>

$$[0, x_n] = (0, x] \cup (x, x_n]$$

$$(x, x_n] = \cdots \cup (x_{n+2}, x_{n+1}] \cup (x_{n+1}, x_n]$$

$$= \bigcup_{k=1}^{\infty} (x_{n+k}, x_n]$$

$$\mu(x, x_n] = \sum_{k=1}^{\infty} \mu(x_{n+k}, x_n]$$

$$F(x_n) - F(x) = \sum_{k=1}^{\infty} \left(F(x_n) - F(x_{n+k}) \right]$$

$$= \lim_{m \to \infty} \left(\sum_{k=1}^{m} F(x_n) - F(x_{n+k}) \right.$$

$$= \lim_{m \to \infty} \Big(F(x_n) - \cancel{F(x_{n+1})} + \cancel{F(x_{n+1})} - \cancel{F(x_{n+2})} + \cancel{F} - \cancel{F}(x_{n+m})$$

$$= \lim_{m \to \infty} \Big(F(x_n) - F(x_{n+m})$$

$$\cancel{F(x_n)} \neq F(x) = \cancel{F(x_n)}$$

$$\neq \lim_{m \to \infty} \left(F(x_{n+m}) \right)$$

$$\lim_{m \to \infty} F(x_{n+m}) = F(x)$$

F is continuous from the right at x, $x \geq 0$