Lecture 7

Measure and Integration

1/12/10

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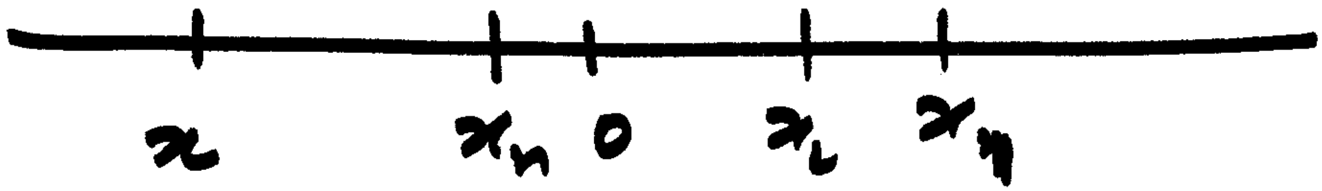
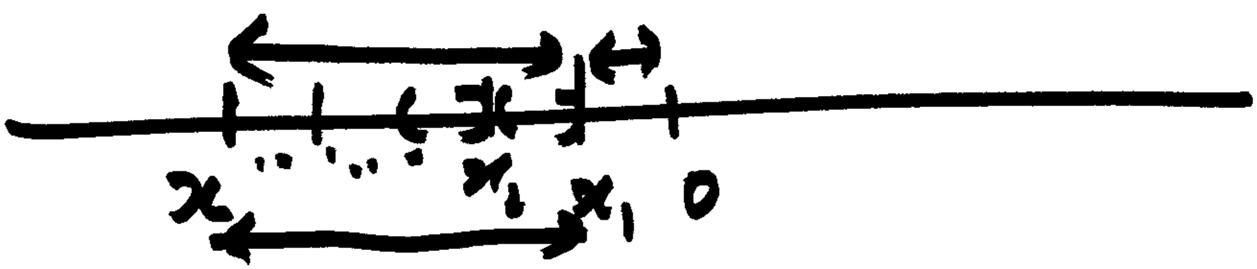
To show F is right continuous at x, $x < 0$. ①



Let $x_n \in \mathbb{R}$, $\underline{x_n \downarrow x}$

w.l.o.g. assume $x_n > 0 \ \forall n$



$$(x, 0] = (x, x_1] \cup (x_1, 0]$$

$$= \left(\bigcup_{n=1}^{\infty} (x_{n+1}, x_n] \right) \cup (x_1, 0]$$

μ c.a $\Rightarrow$

$$\mu(x,0] = \sum_{n=1}^{\infty} \mu(x_{n+1}, x_n] + \mu(x_1, 0]$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(x_{n+1}, x_n] + \mu(x_1, 0]$$

$$-F(x) = \lim_{k\to\infty} \left[\sum_{n=1}^{k} F(x_n) - F(x_{n+1})\right] - F(x_1)$$

$$= \lim_{k\to\infty} \left[\begin{array}{l} F(x_1) - \cancel{F(x_2)} \\ + \cancel{F(x_2)} - \cancel{F(x_3)} \\ \vdots \\ + \cancel{F(x_n)} - F(x_{n+1}) \end{array} \right] - F(x_1)$$

$$= \lim_{k\to\infty} \left[\cancel{F(x_1)} - F(x_{k+1})\right] - \cancel{F(x_1)}$$

③

$$F(x) = \lim_{k \to \infty} F(x_{k+1})$$

$\Longrightarrow$ F is right continuous at x, $x<0$.

Hence F is right cont. $\forall x$.

$$\mu : \tilde{\mathcal{I}} \longrightarrow [0, \infty] \text{ is c.a.}$$

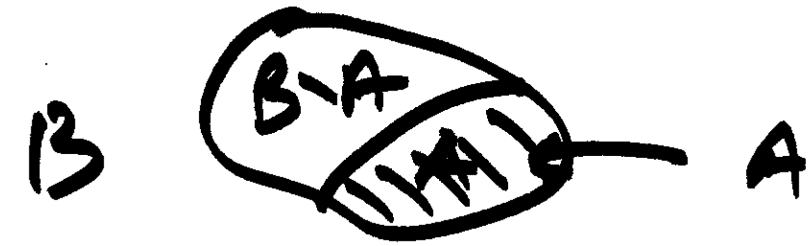
$$\mu(a,b] < +\infty \quad \forall\, a, b \in \mathbb{R}$$

$\Longrightarrow \exists\, F : \mathbb{R} \longrightarrow \mathbb{R}$, nondec, right cont

s.t. $\mu(a,b] = F(b) - F(a)$.

$A, B \in \mathcal{A}$

Given $A \subseteq B$, $\mu(B) < +\infty$.



$$B = A \cup (B \setminus A)$$

μ f.a. $\Longrightarrow \mu(B) = \mu(A) + \mu(B \setminus A)$

$\Longrightarrow \mu(B) \geq \mu(A)$ ✓

$\Longrightarrow \mu(A) \leq \mu(B) < +\infty$

$\Longrightarrow \mu(B) - \mu(A) = \mu(B \setminus A)$

Assume μ is countably additive. (5)

To show μ is finitely additive and countably sub-additive.

let $\underline{A} = \bigsqcup_{i=1}^{n} A_i$, $A_i \in \mathcal{A}$

$$= \bigsqcup_{i=1}^{\infty} A_i, \quad A_i = \phi \text{ if } i > n$$

$$\xRightarrow{c.a} \mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$$

$$= \sum_{i=1}^{n} \mu(A_i)$$

$\Rightarrow \mu$ f.a.

$$\begin{cases} A \subseteq \bigcup_{i=1}^{\infty} A_i, \quad A_i \in \mathcal{A} \\ \bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i, \end{cases}$$

$$B_1 = A_1$$

$$\vdots$$

$$B_n = \underline{A_n} \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$$

$$\vdots$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$\leq \sum_{i=1}^{n} \mu(A_i)$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

$$A \subseteq \bigcup_{i=1}^{\infty} A_i$$

$$\Rightarrow A = \bigcup_{i=1}^{\infty} (A \cap A_i)$$

$$\Rightarrow \underline{\mu(A)} = \mu\left(\bigcup_{i=1}^{\infty} (A \cap A_i)\right)$$

$$\leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$\leq \cancel{\mu} \underline{\sum_{i=1}^{\infty} \mu(A_i)}$$

<u>Assume</u> μ is finitely additive and μ is <u>countably</u> <u>subadditive</u>.

<u>To show</u> μ is countably additive.

Pf let $A \in \mathcal{A}$, $A = \bigsqcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}$

$$\xrightarrow{\text{C.S.A}} \quad \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

<u>To show</u> $\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$ ✓

<u>Enough to show</u> $\mu(A) \geq \sum_{i=1}^{n} \mu(A_i) \quad \forall n$

Note

$$A = \bigsqcup_{i=1}^{\infty} A_i$$

$$\Longrightarrow \forall n \quad \bigsqcup_{i=1}^{n} A_i \subseteq A$$

μ f.a ($\Rightarrow$ μ monoton)

$$\Longrightarrow \mu\left(\bigsqcup_{i=1}^{n} A_i\right) \leq \mu(A) \quad \forall n$$

$$\overset{\text{f.a.}}{\Longrightarrow} \sum_{i=1}^{n} \mu(A_i) \leq \mu(A) \quad \forall n$$

$$\Longrightarrow \sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A)$$

Let μ be countably additive

<u>To show</u> Let $A \in \mathcal{A}$, $A_n \in \mathcal{A}$,

$$A_n \subseteq A_{n+1}, \quad A = \bigcup_{n=1}^{\infty} A_n$$

$$\Longrightarrow \quad \mu(A) = \lim_{n \to \infty} \mu(A_n) \ ?$$

Pf. Let $B_n := A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i \right)$, $n \geq 1$

Then $B_n \in \mathcal{A}$ $\forall n$, $B_n \cap B_m = \phi$

$$A = \bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} B_n$$

$$\overset{c.a.}{\Longrightarrow} \quad \mu(A) = \mu\left(\bigsqcup_{n=1}^{\infty} B_n \right)$$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n)$$

$$= \lim_{k\to\infty} \left(\mu\left(\bigcup_{n=1}^{k} B_n\right)\right)$$

$$= \lim_{k\to\infty} \left(\mu\left(\bigcup_{n=1}^{k} A_n\right)\right)$$

$$= \lim_{k\to\infty} \left(\mu(A_k)\right)$$

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$\Longleftarrow$ μ has the given property

To show μ is c.a., i.e.

$$A = \bigsqcup_{n=1}^{\infty} A_n \quad , \quad A, A_n \in \mathcal{A}.$$

$$= \bigcup_{k=1}^{\infty} \underbrace{\left(\bigsqcup_{n=1}^{k} A_n \right)}_{B_k}$$

$$\text{Given hypothesis} \Longrightarrow \quad \mu(A) = \lim_{k \to \infty} \mu(B_k)$$

$$= \lim_{k \to \infty} \mu\left(\bigsqcup_{n=1}^{k} A_n \right)$$

$$= \lim_{k \to \infty} \left(\sum_{n=1}^{k} \mu(A_n) \right)$$

$$= \sum_{1}^{\infty} \mu(A_n).$$

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$\underline{\mu \text{ c.a.}}$ and

$A_n \downarrow A$:

To show $\mu(A) = \lim_{n\to\infty} \mu(A_n)$

Define $B_n = X \setminus A_n \ \forall n.$

Then $B_n \in \mathcal{A}$, $B_n \uparrow X \setminus A$

$$\Longrightarrow \quad \mu(X \setminus A) = \lim_{n\to\infty} \mu(X \setminus B_n)$$

$$\begin{array}{ccc} \mu(X\setminus A) & = & \lim_{n\to\infty}\mu(X\setminus B_n) \\ \| & & \| \\ \cancel{\mu(X)} - \mu(A) & & \cancel{\mu(X)} \setminus \mu(B_n) \end{array}$$

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

μ has given prop.

To show μ is c.a

$$A = \bigsqcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} A_i \right)$$

$$X \setminus A = \bigcap_{i=1}^{\infty} \left(X \setminus \bigcup_{i=1}^{n} A_i \right)$$

$$= \bigcap_{i=1}^{n} \left(B_n \right)$$

$$\mu(X \setminus A) = \lim_{n \to \infty} \mu(B_n)$$

$$\|$$

$$\cancel{\mu(X)} - \underline{\mu(A)} = \cancel{\mu(X)} - \lim_{n \to \infty} \sum_{i=1}^{n} \mu(A_n)$$

$$= \sum_{1}^{\infty} \mu(A_n)$$