Lecture 14

Measure and Integration

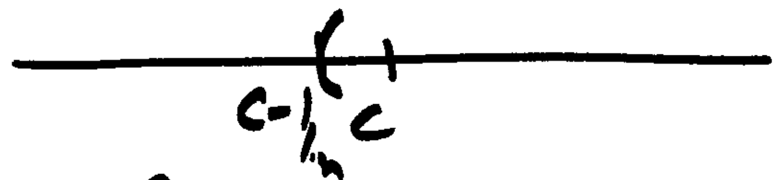
I.K. Rana

22/12/10

Given $\underline{f^{-1}(c,+\infty]} \in \mathcal{S} \quad \forall \; c \in \mathbb{R}$

$$f^{-1}(c,+\infty] = \{x \in X \mid f(x) \in (c,+\infty]\}$$

$f^{-1}[c,+\infty] \in \mathcal{S}$?



<u>Note</u>

$$[c,+\infty] = \bigcap_{n=1}^{\infty} (c-\tfrac{1}{n},+\infty]$$

$$\Longrightarrow \quad f^{-1}[c,+\infty] = f^{-1}\Big(\bigcap_{n=1}^{\infty} (c-\tfrac{1}{n},+\infty]\Big)$$

$$= \bigcap_{n=1}^{\infty} \underbrace{f^{-1}(c-\tfrac{1}{n},+\infty]}_{\in \mathcal{S}}$$

$$\in \mathcal{S}$$

(i) $\Longrightarrow$ (ii)

(ii) $f^{-1}[c,+\infty] \in S \quad \forall\, c \in \mathbb{R}$

$$(c,+\infty] = \bigcup_{n=1}^{\infty} \left[c+\tfrac{1}{n},+\infty\right]$$

$$f^{-1}\big((c,+\infty]\big) = f^{-1}\Big(\bigcup_{n=1}^{\infty}\left[c+\tfrac{1}{n},+\infty\right]\Big)$$

$$= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}\left(\left[c+\tfrac{1}{n},+\infty\right]\right)}_{\in S}$$

$$\Longrightarrow \quad f^{-1}\big((c,+\infty]\big) \in S$$

Hence $\underline{(ii) \Longrightarrow (i)}$

(ii) $f^{-1}([c,+\infty]) \in S \quad \forall\, c \in \mathbb{R}$

$\updownarrow$

$f^{-1}(-\infty, c) \in S \,? \quad \forall\, c \in \mathbb{R}$

$\Longleftrightarrow \quad X \setminus (f^{-1}[c,+\infty]) \in S$

$\qquad\qquad \|$

$f^{-1}[\mathbb{R}^* \setminus [c,+\infty]]$

$\qquad\qquad \|$

$f^{-1}([-\infty, c)) \in S$

(ii) $\Longrightarrow$ (iii)

(iii) $\cancel{f^{-1}(c}$,

$$f^{-1}([-\infty, c)) \in S \quad \forall \; c \in \mathbb{R}$$

<u>Note</u>

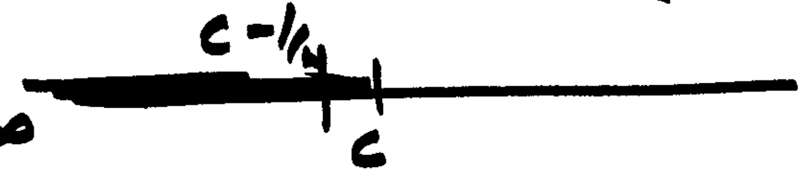
$$[-\infty, c] = \bigcap_{n=1}^{\infty} [-\infty, c + \tfrac{1}{n})$$

$$\Rightarrow \quad f^{-1}[-\infty, c] = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, c + \tfrac{1}{n}))$$

$$\in S$$

<u>(iii) $\Longrightarrow$ (iv)</u>

Given $f^{-1}([-\infty, c]) \in \Sigma \quad \forall c \in \mathbb{R}$



$$[-\infty, c) = \bigcup_{n=1}^{\infty} [-\infty, c-\tfrac{1}{n}]$$

$$f^{-1}[-\infty, c) = f^{-1}\Big(\bigcup_{n=1}^{\infty} [-\infty, c-\tfrac{1}{n}]\Big)$$

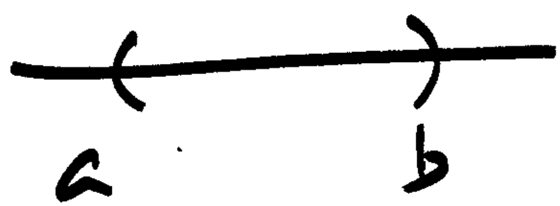
$$= \bigcup_{n=1}^{\infty} \big(\underline{f^{-1}[-\infty, c-\tfrac{1}{n}]}\big)$$

$$\in \Sigma$$

<u>(iv) $\Rightarrow$ (iii)</u>

Assume any one of (i) – (IV)
(hence all)

$$f^{-1}(I) \in \Sigma \quad \forall \quad I = \begin{cases} (c, +\infty] \\ [c, +\infty] \\ [-\infty, c) \\ [-\infty, c] \end{cases}$$



$$(a,b) = (-\infty, b) \cap (a, +\infty)$$

$$f^{-1}(a,b) = f^{-1}(-\infty, b) \cap f^{-1}(a, +\infty)$$

$$\in \Sigma$$

$$[a,b] = [-\infty, b] \cap [a, +\infty]$$

$\Rightarrow f^{-1}(I) \in S \ \forall$ interval

Any open set in $\mathbb{R}$, say U,

$$U = \bigcup_{j=1}^{\infty} I_j, \quad I_j\text{'s open}$$

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{j=1}^{\infty} I_j\Big)$$

$$= \bigcup_{j=1}^{\infty} f^{-1}(I_j) \in S$$

<u>Consider</u> $\mathcal{A} = \{E \in \mathcal{B}_{\mathbb{R}} \mid f^{-1}(E) \in S\}$

Then Open sets $\subseteq \mathcal{A}$.

and $\mathcal{A}$ is a σ-algebra

(i) $\phi, \mathbb{R} \in \mathcal{A}$

(ii) $E \in \mathcal{A} \Rightarrow f^{-1}(E) \in S$

$\Rightarrow (f^{-1}(E))^c \in S$

$\Rightarrow f^{-1}(E^c) \in S$

$\Rightarrow E^c \in \mathcal{A}$

(iii) $E_n \in \mathcal{A} \Rightarrow f^{-1}(E_n) \in S$

$\Rightarrow \bigcup_{n=1}^{\infty} f^{-1}(E_n) \in S$

$\Rightarrow f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) \in S$

$\Rightarrow \bigcup E_n \in \mathcal{A}.$

N.B

$$+\infty = \bigcap_{n=1}^{\infty} (n, +\infty]$$

$$f^{-1}(+\infty) = \bigcap_{n=1}^{\infty} f^{-1}((n, +\infty])$$

$$\in \mathcal{S}$$

$$-\infty = \bigcap_{n=1}^{\infty} [-\infty, -n]$$

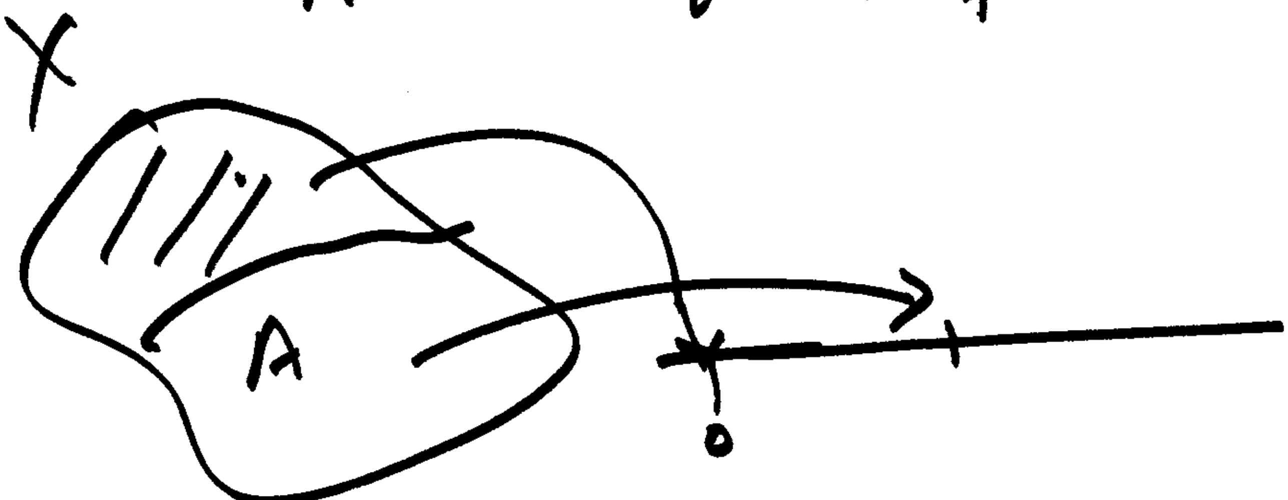
$$\Rightarrow f^{-1}(-\infty) \in \mathcal{S}.$$

X any set

$A \subseteq X$

$\chi_A : X \longrightarrow \{0,1\}$

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$



Characteristic fn.
Indicator fn. of A

$X, \mathcal{S}$

$$\chi_A : X \longrightarrow \mathbb{R}^*$$

<u>Suppose</u> χ_A is measurable

$$\Longrightarrow \quad \chi_A^{-1}(\{1\}) \in \mathcal{S}$$

$$\chi_A^{-1}(\{1\}) = A$$

<u>Conversely</u>: if $A \in \mathcal{S}$, χ_A is mble.

$$(\chi_A^{-1})(I) = \begin{cases} \emptyset & \text{if } 0,1 \notin I \\ A & \text{if } 0 \notin I,\ 1 \in I \\ A^c & \text{if } 0 \in I,\ 1 \notin I \\ X. & \text{if } 0,1 \in I \end{cases}$$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}$$

<u>Note</u>

$$s^{-1}(I) = \left\{ \bigcup_{i : a_i \in I} A_i \right.$$

If $A_i \in \mathcal{S} \ \forall i \implies s^{-1}(I) \in \mathcal{S}$

$\implies s$ is measurable

$\Longleftarrow$ If s is measurable

$$s^{-1}(\{a_i\}) = A_i \in \mathcal{S}$$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}$$

$$\alpha s = \sum_{i=1}^{n} (\alpha a_i) \chi_{A_i}$$

$$s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}, \qquad \bigsqcup A_i = X$$

$$s_2 = \sum_{j=1}^{m} b_j \chi_{B_j} \qquad \bigsqcup B_j = X$$

$$s_1 = \sum_{i=1}^{n} a_i \chi_{\bigcup_j (A_i \cap B_j)} \qquad \chi_{A \sqcup B} = \chi_A + \chi_B$$

$$s_1 = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{A_i \cap B_j}$$

$$= \sum_{i} \sum_{j} a_i \, \chi_{A_i \cap B_j}$$

$$s_2 = \sum_{j=1}^{m} b_j \chi_{B_j} = \sum_{i} \sum_{j} b_j \, \chi_{A_i \cap B_j}$$

$$\underline{s_1 + s_2} = \sum_{i} \sum_{j} (a_i + b_j) \, \chi_{A_i \cap B_j}$$

$$A_i \in \mathcal{S}, \; B_j \in \mathcal{S}$$

$$\Rightarrow A_i \cap B_j \in \mathcal{S}$$

$\Rightarrow s_1 + s_2$ is measurable.

$E \in \Sigma$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}$$

$$s \cdot \chi_E = \sum_{i=1}^{n} a_i (\chi_{A_i} \cdot \chi_E)$$

$$(\chi_{A_i} \cdot \chi_E = \chi_{A_i \cap E})$$

$$s \chi_E = \sum a_i \chi_{A_i \cap E}$$

$A_i \cap E \in \Sigma.$

$\Longrightarrow s \chi_E$ is measurable.

$$s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}$$

$$s_2 = \sum_{j=1}^{m} b_j \chi_{B_j}$$

$$s_1 s_2 = \left(\sum_{j=1}^{n} a_i \chi_{A_i}\right)\left(\sum_{j=1}^{m} b_j \chi_{B_j}\right)$$

$$= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} b_j \chi_{A_i} \chi_{B_j}\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \chi_{\underline{A_i \cap B_j}}$$

$s_1 s_2$ is measurable

$$s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad A_i \in \mathcal{S}$$

$$s_2 = \sum_{j=1}^{m} b_j \chi_{B_j}, \quad B_j \in \mathcal{S}$$

$$s_1 = \sum_i \sum_j a_i \chi_{A_i \cap B_j}$$

$$s_2 = \sum_i \sum_j b_j \chi_{A_i \cap B_j}$$

$$s_1 \vee s_2 = \sum_i \sum_j \max\{a_i, b_j\} \chi_{\underline{A_i \cap B_j}}$$

$$\Rightarrow s_1 \vee s_2 \in \mathcal{S}$$

$$(s_1 \wedge s_2)(x) := \min\{s_1(x), s_2(x)\}$$

$$= \sum_i \sum_j \min\{a_i, b_j\} \chi_{A_i \cap B_j}$$

$\Rightarrow$ $s_1 \wedge s_2$ is a measurable fn.

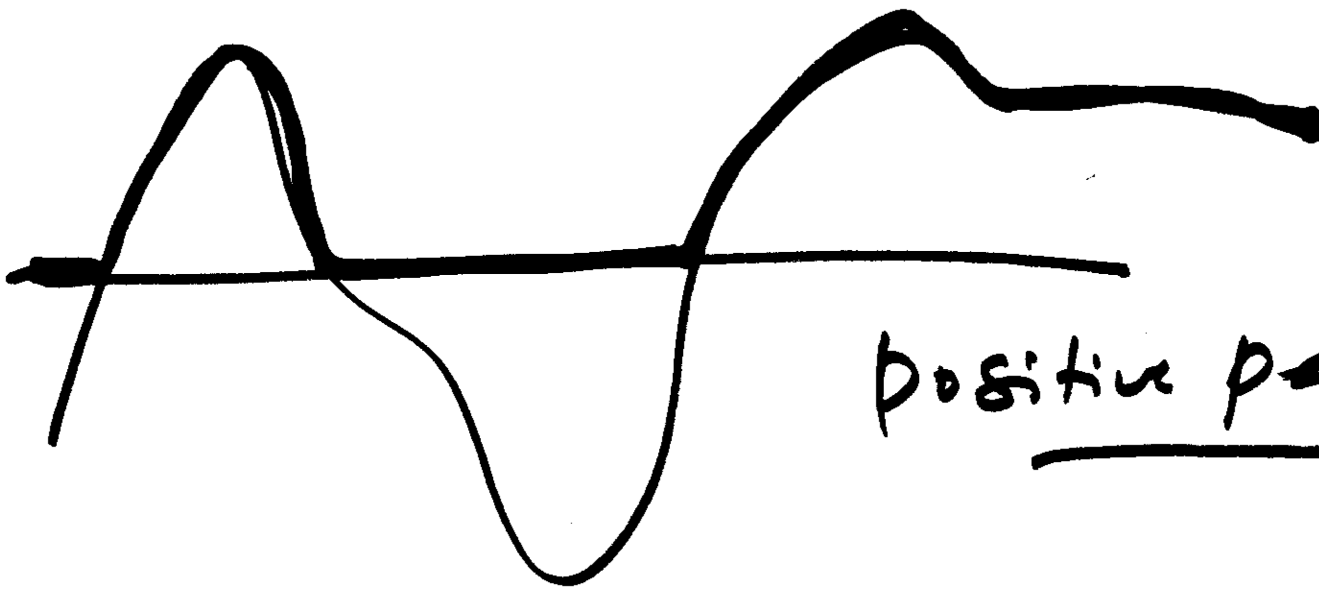
$$s = \sum_{i=1}^{n} a_i \chi_{A_i}$$

$$|s|(x) := |s(x)|$$

$$|s| = \sum_{i=1}^{n} |a_i| \chi_{A_i}$$

$$f : X \longrightarrow \mathbb{R}^*$$

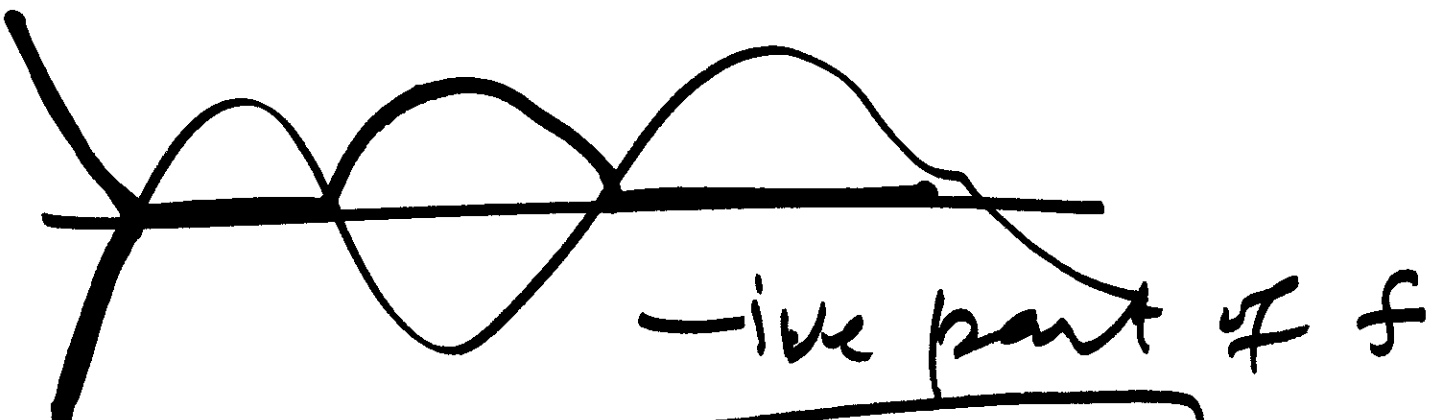
Define
$$f^+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$



positive part

$$f : X \longrightarrow \mathbb{R}^*$$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) > 0 \\ -f(x) & \text{if } f(x) \le 0 \end{cases}$$



<u>Note</u>

$$\boxed{f = f^+ - f^-}$$

$$|f| = f^+ + f^-$$

$$f^+ = \max\{f(x), 0\}$$
$$f^- = \max\{-f(x), 0\}$$