Prof. I.K. Rana
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Lecture 17

Measure & Integration

I.K Rana

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$$\int \chi_A \, d\mu = 1 \times \mu(A)$$

μ a measure on subsets of X

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let $s \in \mathbb{I}_0^+$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{j=1}^{m} b_j \chi_{B_j}$$

where $A_i \in \mathcal{S}$, $B_j \in \mathcal{S}$

$\bigcup_{i=1}^{n} A_i = X$, $\bigcup B_j = X$

$A_i \cap A_j = \phi$, $B_i \cap B_j = \phi$

$\int s \, d\mu$ is well defined:

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j) \ ?$$

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{i=1}^{n} a_i \mu\left(\bigcup_{j=1}^{m} (A_i \cap B_j)\right)$$

$$= \sum_{i=1}^{n} a_i \left[\sum_{j=1}^{m} \mu(A_i \cap B_j)\right]$$

Similarly

$$\sum_{j=1}^{m} b_j \mu(B_j) = \sum_{j=1}^{m} b_j \left[\sum_{i=1}^{n} \mu(A_i \cap B_j)\right] \quad (2)$$

Note that if $x \in A_i \cap B_j$

then $s(x) = a_i = b_j$

if $x \notin A_i \cap B_j$, $s(x) = 0$

$\Rightarrow$ from 1 and 2

$$\sum_{i=1}^{n} a_i \, \mu(A_i) = \sum_{j=1}^{m} b_j \, \mu(B_j)$$

i.e. $\int s \, d\mu =$ is well defined

$s \in \mathbb{I}_0^+,$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad \bigsqcup_i A_i = X$$

$\alpha \in \mathbb{R}^*, \quad \alpha \geq 0,$

$$\alpha s = \sum_{i=1}^{n} (\alpha a_i) \chi_{A_i}, \quad \bigsqcup A_i = X$$

$$\int (\alpha s) \, d\mu = \sum_{i=1}^{n} (\alpha a_i) \mu(A_i)$$

$$= \alpha \left(\sum_{i=1}^{n} a_i \mu(A_i) \right)$$

$$= \alpha \int s \, d\mu$$

$s_1, s_2 \in \mathbb{I}_0^+$

let $s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}$, $\quad \bigsqcup A_i = X$

$s_2 = \sum_{j=1}^{m} b_j \chi_{B_j}$, $\quad \bigsqcup B_j = X$

$$\left.\begin{aligned} s_1 &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \chi_{A_i \cap B_j} \\ s_2 &= \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \chi_{A_i \cap B_j} \end{aligned}\right\} \bigsqcup_{i,j} (A_i \cap B_j) = X$$

$$s_1 + s_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i + b_j) \chi_{A_i \cap B_j}$$

$$
\begin{aligned}
\int (s_1 + s_2)\, d\mu &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j)\, \mu(A_i \cap B_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, \mu(A_i \cap B_j) \\
&\qquad + \sum_{i=1}^{n} \sum_{j=1}^{m} b_j\, \mu(A_i \cap B_j) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(A_i \cap B_j) \\
&\qquad + \sum_{j=1}^{m} b_j \left(\sum_{i=1}^{n} \mu(A_i \cap B_j) \right) \\
&= \sum_{i=1}^{n} a_i\, \mu(A_i) + \sum_{j=1}^{m} b_j\, \mu(B_j) \\
&= \int s_1\, d\mu + \int s_2\, d\mu.
\end{aligned}
$$

$$s \in \mathbb{L}_0^+$$

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad \sqcup A_i = X$$

$E \in \mathcal{S}$.

$$s \cdot \chi_E = \sum_{i=1}^{n} a_i \chi_{A_i} \chi_E$$

$$= \sum_{i=1}^{n} a_i \chi_{A_i \cap E}, \quad \bigcup_{i=1}^{n} (A_i \cap E) = E$$

$$\nu(E) := \int s \chi_E \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E)$$

$$\nu(E) = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E)$$

$\bigsqcup A_i = X$

<u>Claim</u> ν is a measure.

(i) $\nu(\phi) = 0$.

(ii) ν - is countably additive:

let $E = \bigsqcup_{j=1}^{\infty} E_j$.

To show $\nu(E) = \sum_{j=1}^{\infty} \nu(E_j)$?

$$\nu(E) = \sum_{i=1}^{n} a_i \mu(A_i \cap E)$$

$$= \sum_{i=1}^{n} a_i \mu\left(A_i \cap \left(\bigsqcup_{j=1}^{\infty} E_j\right)\right)$$

$$= \sum_{i=1}^{n} a_i \mu\left(\bigsqcup_{j=1}^{\infty} (A_i \cap E_j)\right)$$

$$= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{\infty} \mu(A_i \cap E_j)\right)$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} a_i \mu(A_i \cap E_j)\right)$$

$$= \sum_{j=1}^{\infty} \nu(E_j)$$

$$\nu(E) = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E)$$

$$\bigsqcup A_i = X$$

If $\mu(E) = 0$

$$\Rightarrow \mu(A_i \cap E) = 0 \quad (\because A_i \cap E \subseteq E)$$

$$\Rightarrow \nu(E) = 0$$

$$\underline{\underline{\mu(E) = 0 \Rightarrow \nu(E) = 0}}$$

$$\boxed{\int s \chi_E \, d\mu = \int_E s \, d\mu}$$

Integral of s over E.

$$s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad s_2 = \sum_{j=1}^{m} b_j \chi_{B_j}$$

$$s_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \chi_{A_i \cap B_j}, \quad s_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \chi_{A_i \cap B_j}$$

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (A_i \cap B_j) = X$$

$$s_1 \geq s_2 \Rightarrow a_i \geq b_j \text{ if } x \in A_i \cap B_j$$

$$\int s_1 \, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \, \mu(A_i \cap B_j)$$

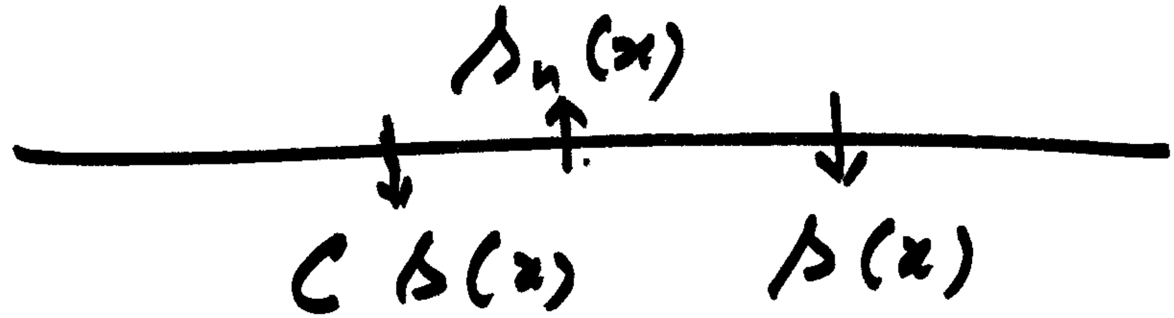
$$\geq \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \, \mu(A_i \cap B_j)$$

$$= \int s_2 \, d\mu$$

Let $\underline{0 < c < 1}$ be fixed.

Then

$$c\, s(x) < s(x)$$



$$s_n(x) \uparrow s(x)$$

Let $B_n = \{x \mid s_n(x) > c\, s(x)\}$

$\Rightarrow$ N.B. $B_n \subseteq B_{n+1}$ $\forall n$

$\Rightarrow$ $B_n \uparrow$, $B_n \in \mathcal{S}$

$$s_1 \wedge s_2 = \min\{s_1, s_2\}$$

$$\begin{aligned} s_1 \wedge s_2 &\leq s_1 \leq s_1 \vee s_2 \\ &\leq s_2 \leq s_1 \vee s_2 \end{aligned}$$

$$\int (s_1 \wedge s_2)\, d\mu \leq \int s_i \, d\mu \leq \int (s_1 \vee s_2)\, d\mu$$

$$i = 1, 2$$

$s_n \in \mathbb{L}_0^+$, $s_n \uparrow s \in \mathbb{L}_0^+$

$$\Longrightarrow \quad \int s\,d\mu = \lim_{n\to\infty} \int s_n\,d\mu \ ?$$

Pf: <u>Note</u> $s_n \uparrow s$

$$\Rightarrow \forall n \ s_n(x) \le s(x) \quad \forall x \in X$$

$$\Rightarrow \quad \int s_n\,d\mu \le \int s\,d\mu \quad \forall n$$

$$\Rightarrow \quad \lim_{n\to\infty} \int s_n\,d\mu \le \int s\,d\mu \quad \text{--- (1)}$$

$$\bigcup_{n=1}^{\infty} B_n = X$$

$$\left[\because \quad \forall\, x \in X, \ \exists\, n_0 \text{ s.t. } s_{n_0}(x) > c\, s(x) \right]$$

$$\nu\,\cancel{\mu}(X) = \lim_{n \to \infty} \cancel{\mu}(B_n) \leftarrow$$

<u>Thus</u>

$$\int c\, s(x)\, d\mu(x) = \int_{B_n} c\, s(x)\, d\mu(x)$$

$$< \int_{B_n} s_n(x)\, d\mu(x)$$

$$\int c\, s(x)\, d\mu(x) \leq \int_X s_n(x)\, d\mu(x) \quad \forall n$$

$$\implies \int c\, s(x)\, d\mu(x) \leq \lim_{n\to\infty} \int s_n\, d\mu$$

$$\forall\ 0 < c < 1$$

$$\implies \int s\, d\mu \leq \lim_{n\to\infty} \int s_n\, d\mu \qquad (2)$$

$$(1) + (2) \implies \int s\, d\mu = \lim_{n\to\infty} \int s_n\, d\mu$$