Lecture no. 2

Measure Theory

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$f, g \in L_1(\mu)$, i.e.

$$\int |f|\,d\mu < +\infty, \quad \int |g|\,d\mu < \infty.$$

$$|f+g| \leq |f| + |g|$$

$$\Longrightarrow \int |f+g|\,d\mu \leq \int (|f| + |g|)\,d\mu$$

$$= \int |f|\,d\mu + \int |g|\,d\mu$$

$$< \infty.$$

$$\Longrightarrow f+g \in L_1(\mu)$$

$f, g \in L_1(\mu)$

$$\Longrightarrow \quad \int f^+ d\mu < +\infty, \ \int f^- d\mu < +\infty$$

$$\int g^+ d\mu < +\infty, \ \int g^- d\mu < +\infty$$

To Show

$$\int (f+g)^+ d\mu < +\infty \ ?$$

$$\int (f+g)^- d\mu < +\infty \ ?$$

$$f+g = (f+g)^+ - (f+g)^-$$

Also

$$f+g = f^+ - f^- + g^+ - g^-$$

$$(f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$$

$$\Rightarrow \quad (f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-$$

$$\Rightarrow \quad \int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu$$

$$= \int f^+ d\mu + \int g^+ d\mu + \int (f+g)^- d\mu$$

$$\Longrightarrow \int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu$$

$$= \int f^+ \, d\mu - \int f^- \, d\mu$$

$$+ \int g^+ \, d\mu - \int g^- \, d\mu$$

$$\Longrightarrow \int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

$$f \in L_1(\mu), \quad E \in \mathcal{S}$$

$\Longrightarrow$ $\chi_E f$ is measurable

and $$|\chi_E f| = \chi_E |f|$$

$$\Longrightarrow \int |\chi_E f| \, d\mu = \int \chi_E |f| \, d\mu$$
$$\leq \int |f| \, d\mu < +\infty$$

$$\Longrightarrow \tilde{\nu}(E) := \int \chi_E f \, d\mu \in \mathbb{R}.$$

Suppose $\mu(E) = 0$.

Then $\tilde{\nu}(E) = \int \chi_E f \, d\mu$

$$= \underbrace{\int \chi_E f^+ \, d\mu}_{=0} - \underbrace{\int \chi_E f^- \, d\mu}_{=0}$$

$\Longrightarrow \tilde{\nu}(E) = 0.$

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Given

$$\tilde{\nu}(E) = \int \chi_E f \, d\mu = 0 \quad \forall E \in \Sigma$$

To show

$$N = \{x \in X \mid |f(x)| > 0\}$$

$$\mu(N) = 0 \, ? \; \checkmark$$

Consider

$$A_n := \{x \in X \mid f(x) > \tfrac{1}{n}\}$$

$$B_n := \{x \in X \mid f(x) < -\tfrac{1}{n}\}$$

$$N = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

Suppose $\mu(N) > 0$

$\Longrightarrow$ $\exists\, n_0$ such that either

$\mu(A_{n_0}) > 0$

or $\mu(B_{n_0}) > 0$ ✓

<u>If $\mu(A_{n_0}) > 0$,</u>

$$\int_{A_{n_0}} f\, d\mu = \int \chi_{A_{n_0}} f\, d\mu > \int_{A_{n_0}} \frac{1}{n_0}\, d\mu$$

$$> \frac{1}{n_0} \cdot \mu(A_{n_0}) > 0$$

Which is not true.

$\Longrightarrow$ $\mu(N) = 0$

$f \in L_1(\mu)$

$\Longrightarrow \quad |f(x)| < +\infty$ a.e. (x)

$N = \{ x \in X \mid |f(x)| = +\infty \}$

To show $\mu(N) = 0$

~~If not~~ $\cancel{A_n = \{x \in X \mid |f(x)| > n\}}$

$\underline{\int |f|\, d\mu} = \underline{\int_N |f|\, d\mu} + \underline{\int_{N^c} |f|\, d\mu}$

If $\mu(N) > 0$, then

$$\int |f| d\mu > \int_N |f| d\mu$$

$$= +\infty \cdot \mu(N) = +\infty$$

if $\mu(N) > 0$, that is not true as $f \in L_1$.

$E_i \in \Sigma, \quad E_i \cap E_j = \phi$ for $i \neq j$

$$E = \bigcup_{i=1}^{\infty} E_i .$$

<u>Claim</u> $\displaystyle\sum_{i=1}^{\infty} \left(\int_{E_i} f\,d\mu\right)$ is absolutely cgt.

<u>Note</u> $$\left|\int_{E_i} f\,d\mu\right| \leq \int_{E_i} |f|\,d\mu$$

and $$\sum_{i=1}^{\infty} \int_{E_i} |f|\,d\mu = \int_{E} |f|\,d\mu < +\infty$$

$\Rightarrow$ Claim holds.

Hence

$$\int_E f\,d\mu = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\int_{E_i} f\,d\mu\right)$$

$$= \sum_{i=1}^{\infty}\int_{E_i} f\,d\mu$$

<u>Given</u> f_n are mble, for $n \geq 1$

$$\left[\begin{array}{l} |f_n(x)| \leq g(x) \text{ for } \underline{\text{a.e.}}(x), \forall n \\ f_n(x) \longrightarrow f(x) \ \underline{\text{a.e.}} \end{array}\right.$$

<u>Assume</u> $|f_n(x)| \leq g(x), \quad g \in L_1(\mu)$

$$\Longrightarrow \int |f_n(x)| \, d\mu(x) \leq \int g(x) \, d\mu(x) < +\infty$$

$$\Longrightarrow f_n \in L_1(\mu)$$

Also $\underline{|f_n(x)|} \longrightarrow |f(x)|$

$\Rightarrow$ $\cancel{|f_n(x)|}$,

$$|f(x)| \leq g(x) \; \forall x$$

$\Rightarrow$ $\int |f| d\mu < \int g \, d\mu < +\infty$

$\Rightarrow$ $f \in L_1(\mu)$

Note $f_n \to f$, $|f_n| \leq g \in \mathcal{L}$

$\Rightarrow$ $\cancel{\{f_n - g\}_{n\geq 1}}$ $\{g - f_n\}_{n\geq 1}$

is a sequence of mble fns,

and $g - f_n \geq 0$

$$g - f_n \longrightarrow g - f$$

Fatou's lemma $\Longrightarrow$

$$\int \liminf (g - f_n)\, d\mu \le \liminf \left(\int (g - f_n)\, d\mu \right)$$

$$= \int \left[g + \liminf(-f_n) \right] d\mu$$

$$= \int g\, d\mu + \int \left(-\limsup f_n \right) d\mu$$

$$= \int g\, d\mu - \int \cancel{}\, f(x)\, d\mu$$

$$= \int g\, d\mu - \limsup \int f_n\, d\mu$$

$$\cancel{\int g\,d\mu} - \int f\,d\mu \le \cancel{\int g\,d\mu} - \limsup \int f_n\,d\mu$$

$$\Longrightarrow \quad \int f\,d\mu \geq \limsup\left(\int f_n\,d\mu\right) \quad \text{— (1)}$$

Similarly $\{g + f_n\}_{n \geq 1}$

$\underline{\text{Fatou's lemma}}$

$$\Longrightarrow \quad \int f\,d\mu \le \liminf\left(\int f_n\,d\mu\right) \quad \text{— (2)}$$

$$(1) + (2) \Longrightarrow \lim_{n \to \infty} \int f_n\,d\mu = \int f\,d\mu.$$

$\square$