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Lec. No. 22
Date: 31/1/11

Lecture 22

Measure and Integration

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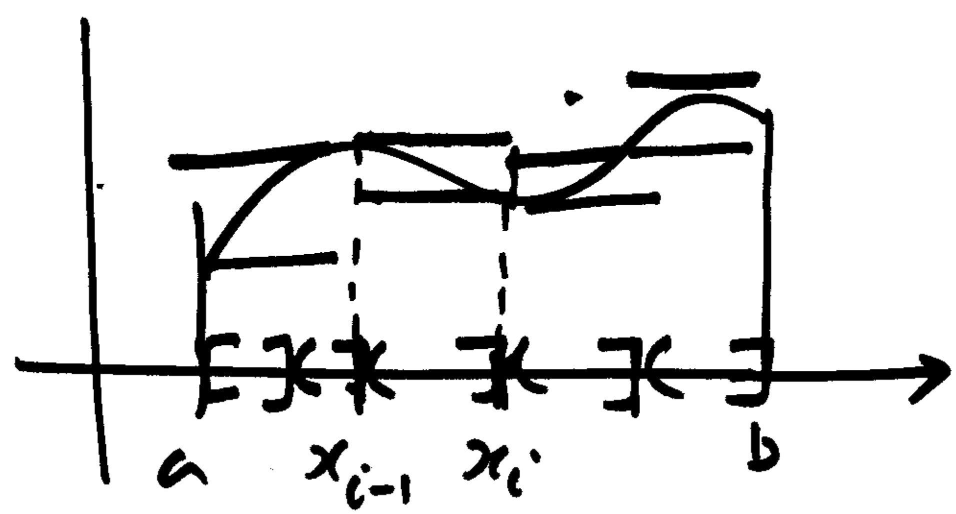
31/1/11

$f : [a,b] \longrightarrow \mathbb{R}$, $\underline{f \in \mathcal{R}[a,b]}$.

$\exists$ a sequence $\{P_n\}_{n \geq 1}$ of refinement partitions of $[a,b]$ such that, $\|P_n\| \to 0$ and

$$U(f, P_n) \downarrow, \quad L(f, P_n) \uparrow$$

$$\lim_{n \to \infty} U(f, P_n) = \int_a^b f(x)\,dx = \lim_{n \to \infty} L(f, P_n).$$



Define $P_n = \{a = x_0 < \cdots < x_{i-1} < x_i \cdots x_n = b\}$

$$M_1 = \sup\{f(x) \mid a \leq x \leq x_1\}$$

$$M_k = \sup\{f(x) \mid x_{i-1} < x \leq x_i\}$$

$$m_k = \inf\{f(x) \mid x_{i-1} < x \leq x_i\}$$

$$m_1 = \inf\{f(x) \mid x_0 \leq x \leq x_1\}$$

$$\phi_n = \sum_{i=2}^{n} m_i \, \chi_{(x_{i-1}, x_i]} + m_1 \chi_{[a, x_1]}$$

$$\psi_n = \sum_{i=2}^{n} M_i \, \chi_{(x_{i-1}, x_i]} + M_1 \chi_{[a, x_1]}$$

$$\phi_n(x) \leq f(x) \leq \psi_n(x)$$

ϕ_n, ψ_n is a Step function.

$$\phi_n \uparrow \text{ and } \psi_n \downarrow$$

Further

$$\int_a^b \phi_n(x)dx = m_1(x_1-a)+\sum_{k=1}^{n} m_k(x_k-x_{k-1})$$
$$= L(f,P_n)$$

$$\int_a^b \psi_n(x)\,dx = M_1\,(x_1-a)+\sum_{k=1}^{n} M_k(x_k-x_{k-1})$$
$$= U(f,P_n)$$

and $$\lim_{n\to\infty}\int_a^b \phi_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b \psi_n(x)dx$$

$\forall n$

$$\int_a^b \phi_n(x)\,dx = \int_{[a,b]} \phi_n \, d\lambda$$

$$\int_a^b \psi_n(x)\,dx = \int_{[a,b]} \psi_n \, d\lambda$$

<u>Consider</u> $\{\psi_n - \phi_n\}_{n\geq 1}$ and apply)
Fatou's lemma:

$$\int_{[a,b]} \liminf_{n\to\infty} (\psi_n - \phi_n)\,d\lambda \leq \liminf \int_{[a,b]} (\psi_n - \phi_n)\,d\lambda = 0$$

$\Rightarrow \quad \liminf (\psi_n - \phi_n)(x) = 0$ a.e. x.

$\Rightarrow \quad \lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \phi_n(x)$ a.e. x.

Since $\phi_n(x) \leq f(x) \leq \psi(x)$

$\Rightarrow \quad \boxed{\lim_{n\to\infty} \psi_n(x) = f(x) = \lim_{n\to\infty} \phi_n(x) \text{ a.e. } x}$

$\Rightarrow$ f is measurable.

$\Rightarrow$ f is Lebesgue integrable ($\because f$ bdd)

__Claim__ $\int_{[a,b]} f \, d\lambda = \int_a^b f(x)\, dx$?

$$\psi_n(x) \longrightarrow f(x) \text{ a.e. } x$$

$$\psi_n \downarrow, \quad \psi_n \in L_1[a,b]$$

LDCTm $\Longrightarrow$

$$\int_{[a,b]} f \, d\lambda = \lim_{n\to\infty} \int_{[a,b]} \psi_n \, d\lambda$$

$$= \lim_{n\to\infty} \int_a^b \psi_n(x)\, dx$$

$$= \lim_{n\to\infty} U(f, P_n)$$

$$= \int_a^b f(x)\, dx.$$

$\square$

$\mathbb{R}^n$ $\quad \underline{x} = (x_1, \dots, x_n)$

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i| \qquad \left(L_1\text{-norm on } \mathbb{R}^n\right)$$

Treat $\underline{f} : [a,b] \longrightarrow \mathbb{R}$

as a vector with as many components as points in $[a,b]$

$x \in [a,b]$ $\quad x^{th}$ component of f

$f(x)$

$$\{f(x) \mid x \in [a,b]\}$$

$$\|f\|_1 = \int |f(x)|\, d\lambda$$

$\{f_n\}_{n\geq 1}$ is Cauchy

$\iff$ $\forall\ \varepsilon > 0$, $\exists\ n_0$ such that

$$\|f_n - f_m\|_1 < \varepsilon \quad \forall\ n, m \geq n_0$$

<u>$\varepsilon = 1$</u>, find n_1 such that

$$\|f_n - f_m\|_1 < \varepsilon = 1 \quad \forall\ n, m \geq n_1$$

i.e. $\|f_n - f_{n_1}\|_1 < 1 \quad \forall\ n \geq n_1$

Suppose $n_1 < n_2 < \dots < n_k$ have been found. Find n_{k+1} such that

$$\|f_n - f_m\| < \frac{1}{2^{k+1}} \ \forall\ n, m \geq n_{k+1}$$

$$(n_{k+1} > n_k)$$

$$\|f_n - f_{n_{k+1}}\| < \frac{1}{2^{k+1}} \ \forall\ n \geq n_{k+1}$$

By induction, $\exists\ n_1 < n_2 < \cdots < n_k < \cdots$

s.t. $\|f_n - f_{n_k}\|_1 < \frac{1}{2^k} \ \forall\ n \geq n_k$

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k} \ \forall\ k.$$

$$\|f_{n_1}\|_1 + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_1 \leq \|f_{n_1}\|_1 + \sum_1^{\infty} \frac{1}{2^j} < +\infty$$

$$f - f_{n_j} = \sum_{k=n_j+1}^{\infty} [f_{n_k} - f_{n_{k-1}}]$$

$$\|f - f_{n_j}\|_1 \leq \sum_{n_j+1}^{\infty} \|f_{n_k} - f_{n_{k-1}}\| \longrightarrow 0 \text{ as } j \to \infty$$

$$\underline{\|f - f_{n_j}\| \longrightarrow 0} \quad j \to \infty$$