Lecture 27

Measure and Integration

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$$\mathcal{P} = \left\{ E \in \mathcal{A} \otimes \mathcal{B} \;\middle|\; \begin{array}{l} x \longmapsto \nu(E_x) \\ y \longmapsto \mu(E_y) \end{array} \right\} \text{ mble}$$

and
$$\int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E_y)\, d\nu(y)$$
$$= (\mu \times \nu)(E)$$

(1) Let $E_n \in \mathcal{P}$, $n \geq 1$, $E_n \uparrow$. To show

$$E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{P}?$$

$x \longmapsto \nu(E_x)$ is $\mathcal{A}$-mble fn.?

$E_n \in \mathcal{P} \Rightarrow$ $x \longmapsto \nu((E_n)_x)$ is mble $\forall n$.

$E_n \uparrow E$, $\quad (E_n)_x \uparrow E_x$ in $\mathcal{B}$

$$\Rightarrow \quad \nu((E_n)_x) \longrightarrow \nu(E_x) \quad — \; (*)$$

$$\Rightarrow \quad x \longmapsto \nu(E_x) \text{ is mble.}$$

Next

$$\int_X \nu(E_x)\, d\mu(x) = (\mu \times \nu)(E) \, ?$$

$(*)$ and monotone convergence thm

$$\Rightarrow \quad \int_X \nu(E_x)\, d\mu(x) = \lim_{n \to \infty} \int_X \nu((E_n)_x)\, d\mu(x)$$

$$\| $$

$$= \lim_{n\to\infty} (\mu\times\nu)(E_n)$$

$$\int_X \nu(E_x)\,d\mu(x) = (\mu\times\nu)(E)$$

$$\Longrightarrow \quad E \in \mathcal{P}$$

<u>Next</u> $E_n \in \mathcal{P}$, $n \geq 1$, $E_n \downarrow E$, i.e.

$$E = \bigcap_{n=1}^{\infty} E_n.$$

Claim $E \in \mathcal{P}$?

$$\left.\begin{aligned} &\textbf{Assume}\quad \mu(X) < +\infty \\ &\nu(Y) < +\infty \\ \Longrightarrow\quad &(\mu \times \nu)(X \times Y) < +\infty \end{aligned}\right\} \oplus$$

$$E_n \in \mathcal{P},\ E_n \downarrow E \Longrightarrow E \in \mathcal{P}\,?$$

$$E_n \downarrow E \Longrightarrow (E_n)_x \downarrow E_x$$

$$\Longrightarrow \nu((E_n)_x) \xrightarrow{\oplus} \nu(E_x)$$

$$\Longrightarrow x \longmapsto \nu(E_x) \text{ is mble.}$$

Note $\mu(X), \nu(Y) < +\infty$, $x \longmapsto \nu((E_n)_x)$ is an integrable function.

$$\nu((E_n)_x) \leq \nu((E_1)_x)$$

and $$\int_X \nu((E_1)_x)\, d\mu(x) \leq \nu(Y)\mu(X) < +\infty$$

Dominated cgce thm

$$\left(\nu((E_n))_x \downarrow \nu(E_x) \right)$$

$$\int_X \nu(E_x)\, d\mu(x) = \lim_{n\to\infty} \int \nu((E_n)_x)\, d\mu(x)$$
$$= \lim_{n\to\infty} (\mu \times \nu)(E_n)$$

$$\int_X \nu(E_x)\,d\mu(x) \overset{(+)}{=} (\mu\times\nu)(E)$$

$\mathcal{P}$ <u>is a monotone class</u>

$$\mathcal{F}(\mathcal{R}) \subseteq \mathcal{P}$$

$$\Longrightarrow \mathcal{M}(\mathcal{F}(\mathcal{R})) \subseteq \mathcal{P}$$

$$\mathcal{M}(\mathcal{F}(\mathcal{R})) = \mathcal{S}(\mathcal{F}(\mathcal{R})) = \mathcal{S}(\mathcal{R})$$

$$\underline{\mathcal{A}\otimes\mathcal{B} = \mathcal{S}(\mathcal{R})}$$

$$E \in \mathcal{A} \otimes \mathcal{B}$$

$$\int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E_y)\, d\nu(y)$$

$$\|$$

$$(\mu \times \nu)(E) = \int_{X \times Y} \chi_E \, d(\mu \times \nu)$$

$$\int_X \left(\int_Y \chi_{E_x}(y)\, d\nu(y) \right) d\mu(x) \qquad \chi_{E_x}(y) = \chi_E(x,y)$$

$$\int_Y \left(\int_X \chi_{E_y}(x)\, d\mu(x) \right) d\nu(y) \qquad \chi_{E_y}(x) = \chi_E(x,y)$$