Lecture 30

Measure and Integration

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$\underline{X}$ $\underline{Y}$

$\underline{\mathcal{C}}$ $\underline{\mathcal{D}}$ — collections of subsets

$\mathcal{C} \times \mathcal{D}$ — subsets of $X \times Y$

$= \{E \times F \mid E \in \mathcal{C}, F \in \mathcal{D}\}$

$\mathcal{S}(\mathcal{C} \times \mathcal{D}) = \mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D})$

if $X = \bigsqcup_{i=1}^{\infty} C_i$, $\mathcal{D} = \bigsqcup_{j=1}^{\infty} D_j$

$$\Rightarrow \quad \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$$

$$\Rightarrow \quad \mathcal{S}(\tilde{\mathcal{I}}) \otimes \mathcal{S}(\tilde{\mathcal{I}}) = \mathcal{S}(\tilde{\mathcal{I}} \times \tilde{\mathcal{I}})$$

$$\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{S}(\bar{\mathcal{I}}) \otimes \mathcal{S}(\bar{\mathcal{I}}) = \mathcal{S}(\mathcal{I} \times \mathcal{I}) = \mathcal{S}(\mathcal{I}^2)$$

$\forall\ E \in \mathcal{B}_{\mathbb{R}^2},\quad E + \underline{x} \in \mathcal{B}_{\mathbb{R}^2}$

$\mathcal{A} = \{ E \in \mathcal{B}_{\mathbb{R}^2} \mid E + \underline{x} \in \mathcal{B}_{\mathbb{R}^2} \}$

<u>Claim</u> (i) Open subsets of $\mathbb{R}^2 \subseteq \mathcal{A}$

(ii) $\mathcal{A}$ is a σ-algebra.

$\Longrightarrow \quad \mathcal{S}(\text{Opensets}) \subseteq \mathcal{A}.$

$\qquad\qquad\qquad \| \\ \qquad\qquad \mathcal{B}_{\mathbb{R}^2}$

$$E \in \mathcal{A} \Rightarrow E + x \in \mathcal{B}_{\mathbb{R}^2}$$

$$\Rightarrow (E + x)^c \in \mathcal{B}_{\mathbb{R}^2}$$

$$\parallel$$

$$E^c + x \in \mathcal{B}_{\mathbb{R}^2}$$

$$\Rightarrow E^c \in \mathcal{A}$$

(iii) $E_i \in \mathcal{A} \Rightarrow E_i + x \in \mathcal{B}_{\mathbb{R}^2}$

$$\Rightarrow \bigcup_i (E_i + x) \in \mathcal{B}_{\mathbb{R}^2}$$

$$\parallel$$

$$\left(\bigcup E_i\right) + x \in \mathcal{B}_{\mathbb{R}^2}$$

$$\Rightarrow \bigcup E_i \in \mathcal{A}$$

let U be open in $\mathbb{R}^2$

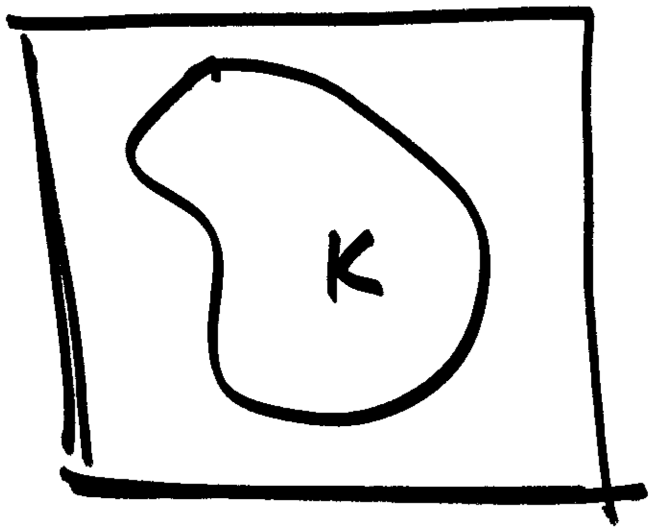
$\Rightarrow$ $U + x \in \mathcal{B}_{\mathbb{R}^2}$

[$\because$ U open $\Rightarrow$ $U + x$ is open]

$y + x \in U + x$, $y \in U$

$\Rightarrow$ $y \in B_\delta(\underline{y}) \subseteq U$

$\Rightarrow$ $\underline{y + x} \in \left(B_\delta(\underline{y}) + x\right) \subseteq \underline{U + x}$



K bdd $\Rightarrow$ $K \subseteq I \times J$

$\Rightarrow \lambda(K) \leq \lambda(I) \times \lambda(J) < +\infty$

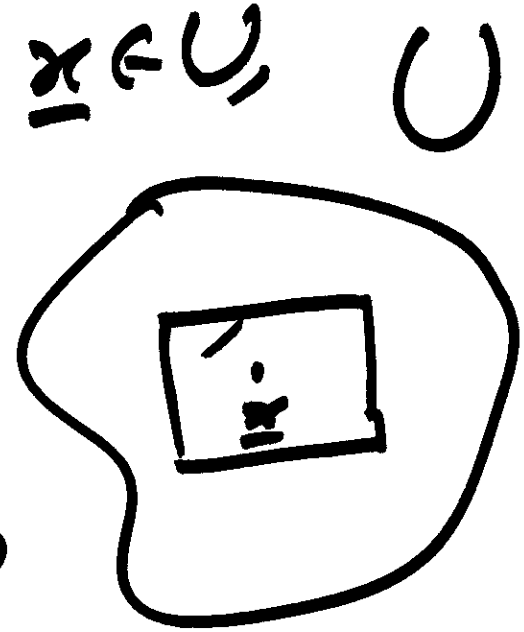
$U \subset \mathbb{R}^2$ is open

and $U \neq \emptyset$, $\underline{x} \in U$,

$\Rightarrow \exists$ a rectangular nbhd

of $\underline{x} \in N \subseteq U$

$\Rightarrow$ $\underline{\lambda(U) \geq \lambda(N) > 0}$



$K \subseteq \mathbb{R}^2$, K compact

K compact $\Rightarrow$ K bounded.

$E \in \mathcal{B}_{\mathbb{R}^2} \Rightarrow E + x \in \mathcal{B}_{\mathbb{R}^2}$

$$\lambda(E+x) = \lambda(E)$$

Define $\mathcal{M} = \{ E \in \mathcal{B}_{\mathbb{R}^2} \mid \lambda(E+x) = \lambda(E) \}$

To show $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{M}$.

(1) Show $\mathcal{M}$ is a monotone class ✓

(2) $\mathcal{M}$ is closed under finite disjoint unions.

(3) $\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$.

(3) + (2) $\Rightarrow$ $\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$ a semi-algebra

$\mathcal{F}(\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}) \subseteq \mathcal{M}$.

$\overset{(1)}{\Longrightarrow}$ $\mathcal{M}(\mathcal{F}(\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}})) \subseteq \mathcal{M}$

$$\begin{gathered} \mathcal{M}(\mathcal{F}(\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}})) \\ \parallel \\ \mathcal{S}(\mathcal{F}(\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}})) \\ \parallel \\ \mathcal{B}_{\mathbb{R}^2} \end{gathered}$$

Proof (1)

$E_n \uparrow E$, $E_n \in \mathcal{M}$.

$\Rightarrow \lambda(E_n + x) = \lambda(E_n) \ \forall n.$

$\Rightarrow$ $\lambda(E_n + x) \to \lambda(E + x)$, $\lambda(E_n) \to \lambda(E)$

$\Rightarrow E \in \mathcal{M}.$

Let $E_1, E_2 \in \mathcal{M}$, $E_1 \cap E_2 = \phi$

$$\Rightarrow \quad \lambda(E_1 + x) = \lambda(E_1)$$

$$\lambda(E_2 + x) = \lambda(E_2)$$

$$E_1 \cap E_2 = \phi \Rightarrow (E_1 + x) \cap (E_2 + x) = \phi$$

$$\Rightarrow \quad \lambda((E_1 + x) \cup (E_2 + x)) = \lambda((E_1 \cup E_2) + x)$$

$$= \lambda(E_1 + x) + \lambda(E_2 + x)$$

$$= \lambda(E_1) + \lambda(E_2)$$

$$= \lambda(E_1 \cup E_2)$$

$$\Rightarrow \quad E_1 \cup E_2 \in \mathcal{M}.$$

$E, F \in \mathcal{B}_{\mathbb{R}}$

To show $E \times F \in \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$

$\underline{x} = (a, b)$

$E + a \in \mathcal{B}_{\mathbb{R}}, \; F + b \in \mathcal{B}_{\mathbb{R}}$

$\lambda(E+a) = \lambda(E), \; \lambda(F+b) = \lambda(F)$

$$(E \times F) + \underline{x} = (E+a) \times (F+b)$$

$$\lambda((E \times F) + \underline{x}) = \lambda(E+a)\,\lambda(F+b)$$

$$= \lambda(E)\,\lambda(F)$$

$$= \lambda(E \times F)$$

$f \geq 0$ mble on $\mathbb{R}^2$

$$\int f(x+y)\, d\lambda_{\mathbb{R}^2}(x) = \int f(x)\, d\lambda_{\mathbb{R}^2}(x) \;?$$

① $f = \chi_E$, $E \in \mathcal{B}_{\mathbb{R}^2}$

$$\int \chi_E(x+y)\, d\lambda_{\mathbb{R}^2}(x) = \lambda_{\mathbb{R}^2}(E-y) = \lambda_{\mathbb{R}^2}(E)$$

$$\int \chi_E(x)\, d\lambda_{\mathbb{R}^2} = \lambda_{\mathbb{R}^2}(E)$$

$\Rightarrow$ (2) Claim holds for

$f =$ non-negative simple measurable fn: $\mathbb{R}^2 \to \mathbb{R}$

(3) $f \geq 0$ mble,

$\Rightarrow \exists\ s_n \uparrow f$,

$$\int f \, d\lambda = \lim_{n\to\infty} \int s_n \, d\lambda$$

$\forall n$ $\int s_n(x+y)\, d\lambda_{\mathbb{R}^2}(x) = \int s_n(x)\, d\lambda_{\mathbb{R}^2}(x)$

MCT $\Downarrow$ $\quad\downarrow$

$$\underline{\int f(x+y)\, d\lambda_{\mathbb{R}^2}(x) = \int f(x)\, d\lambda}$$

$$\int f(x)\, d\lambda_{\mathbb{R}^2}(x) = \int f(-x)\, d\lambda_{\mathbb{R}^2}$$

$f = \chi_E$

$$\lambda_{\mathbb{R}^2}(E) = \lambda_{\mathbb{R}^2}(-E) \checkmark$$

$$-E = \{-\underline{x} \mid \underline{x} \in E\}$$

Defin $\mathcal{A} = \{E \in \mathcal{B}_{\mathbb{R}^2} \mid \lambda(E) = \lambda(-E)$

Show $E \times F \in \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$

then $E \times F \in \mathcal{A}$, and

$\mathcal{A}$ is a σ-algebra. // Ex