Lecture 35

Measure and Integration

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$\underline{x} \in \mathbb{R}^n$, $\underline{x} = (x_1, \ldots, x_n)$
$\underline{y} = (y_1, \ldots, y_n)$

$$\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

$\underline{x}, \underline{y} \in \mathbb{C}^n$

$$\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\| \underline{x} \|^2 = \langle \underline{x}, \underline{x} \rangle$$

$\underline{x} \in \mathbb{R}^n$, $\underline{x} = (x_1, \ldots, x_n)$

$$\|\underline{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$f \in \mathbf{L}_2$, $f(x)$ — x^{th} component of f

$$\|f\|_2 = \left(\int |f(x)|^2 \, d\mu\right)^{1/2}$$

$$L_2(X, \mathcal{S}, \mu) = \{ f : X \longrightarrow \mathbb{C} \mid \int |f|^2 \, d\mu < +\infty$$

$$f \in L_2(X)$$

$$\|f\|_2 = \left(\int |f|^2 \, d\mu \right)^{1/2}$$

(Similar to magnitude in $\mathbb{R}^n$)

$$f, g \in L_2, \quad f \perp g$$

$$\|f+g\|_2^2 = \langle f+g, f+g \rangle$$

$$= \langle f, f \rangle + \langle g, f \rangle$$

$$+ \langle f, g \rangle + \langle g, g \rangle$$

$$= \|f\|^2 + \|g\|^2$$

$$\langle f,f\rangle = \int |f|^2\,d\mu \geq 0$$

$$= 0 \iff |f| = 0 \text{ a.e.}$$

$$\iff f \in L_2,\ f = 0.$$

$$\langle f,g\rangle = \int f\bar{g}\,d\mu = \overline{\left(\int \bar{f} g\,d\mu\right)}$$

$$= \overline{\langle g,f\rangle}$$

$f \in L_p, \; g \in L_q$

$fg \in L_1$

$\int |f \cdot g| \, d\mu \leq \|f\|_p \, \|g\|_q$

$p = 2, \; q = 2 \quad 1/p + 1/q = 1$

$|\langle f, g \rangle| \leq \int |f g| \, d\mu \leq \|f\|_2 \, \|g\|_2$

let $f_n \in S^{\perp}$, $f_n \to f$ in L_2

<u>Claim</u> $f \in S^{\perp}$.

let $h \in S$,

$$\langle f, h \rangle \cancel{= \lim_{n\to\infty} \langle f_n, h \rangle}$$

$$= \lim_{n \to \infty} \langle f_n, h \rangle \text{ ?}$$

$$\left| \langle f, h \rangle - \langle f_n, h \rangle \right|$$

$$= |\langle f - f_n, h \rangle|$$

$$\leq \underbrace{\|f - f_n\|_2}_{\downarrow 0} \|h\|$$

$$\therefore \quad \langle f, h \rangle = \lim_{n\to\infty} \langle f_n, h \rangle$$

$$= 0 \quad \forall h \in S$$

$$\Rightarrow f \in S^\perp$$

S a subset of L_2

$$S^{\perp} = \{ f \in L_2 \mid f \perp h \ \forall h \in S \}$$

$S^{\perp}$ <u>is a subspace</u>

$h, g \in S^{\perp}$, $\alpha, \beta \in \mathbb{C}$, $f \in S$

$$\langle \alpha h + \beta g, f \rangle$$

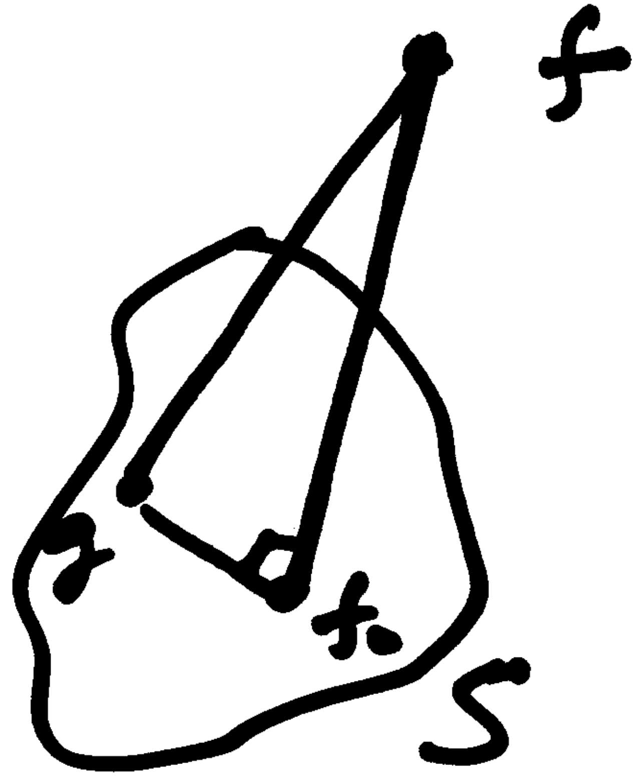
$$= \alpha \underbrace{\langle h, f \rangle}_{= 0} + \beta \underbrace{\langle g, f \rangle}_{= 0}$$

$$= 0$$

$\exists!\ f_0 \in S$

s.t $\alpha = \|f - f_0\|$

$f - f_0 \perp S$



0

$$f \in S_1 \cap S_1^{\perp}$$

$$\Rightarrow \langle f, f \rangle = 0$$

$$\Rightarrow \|f\| = 0$$

$$\Rightarrow f = 0.$$

$$S_1 \cap S_1^{\perp} \subseteq \{0\}$$

$f \in S_1$, $\quad h \in S_1^{\perp}$

$\Rightarrow \langle h, f \rangle = 0$

$\Rightarrow f \in (S_1^{\perp})^{\perp}$

$$S_1 \subseteq (S_1^{\perp})^{\perp}$$

In case $(S_1^{\perp})^{\perp} = S_1$

$\Rightarrow S_1$ is a closed subspace.

Thm $\Longrightarrow$ $\exists\, f_0 \in S_1$ such that

$$f - f_0 \perp S_1$$

$$\Longrightarrow \quad (f - f_0) \in S_1^{\perp} \quad \text{———} \; *$$

Also $f_0 \in S_1 \subseteq (S_1^{\perp})^{\perp}$

$$\Longrightarrow \quad f - f_0 \in (S_1^{\perp})^{\perp} \quad \text{———} \; *$$

$$\Longrightarrow \quad f - f_0 = 0 \Longrightarrow f = f_0$$

$$\Longrightarrow \quad f \in S_1$$

~~**Suppose** $S_1 = (S_1^{\perp})^{\perp}$~~

~~**Claim** S_1 is a closed subspace~~

Suppose S_1 is a closed subspace, then

$$S_1 = (S_1^{\perp})^{\perp} \checkmark$$

let $\exists\ f \in (S_1^{\perp})^{\perp},\ f \notin S_1$

Pythagoras

$\Longrightarrow$

$$\cancel{\|(f_n + g_m)^2 + (g}$$

$$\|(f_n - f_m)^2 + \|(g_n - g_m)^2\|$$

$$= \underbrace{\|(f_n + g_n) - (f_m - g_m)\|^2}_{\longrightarrow 0}$$

$$\Longrightarrow \quad \|f_n - f_m\| \longrightarrow 0, \quad \|g_n - g_m\| \to 0$$

Let $f_n + g_n \in S_1 + S_2$

$$f_n + g_n \longrightarrow f \text{ in } L_2$$

<u>To show</u> $f \in S_1 + S_2$?

$\{f_n + g_n\}_{n \geq 1}$ is Cauchy

$$\|(f_n + g_n) - (f_m + g_m)\| \longrightarrow 0 \quad \text{as } n, m \to \infty$$

<u>Note.</u>

$$f_n - f_m \in S_1 \qquad \Big| \quad S_1 \perp S_2$$
$$g_n - g_m \in S_2$$

S_1, S_2 – closed subspaces

$S_1 \perp S_2$

$\underline{S_1 + S_2}$: $\quad f_1 + g_1 \in S_1 + S_2$

$f_2 + g_2 \in S_1 + S_2$

$\Rightarrow \alpha(f_1 + g_1) + \beta(f_2 + g_2)$

$= \underbrace{(\alpha f_1 + \beta f_2)}_{\in S_1} + \underbrace{(\alpha g_1 + \beta g_2)}_{\in S_2}$

$\Rightarrow\ \in S_1 + S_2$

$\Rightarrow \{f_n\}_{n\geq 1}$ is cauchy

$\Rightarrow \quad f_n \longrightarrow h \in S_1$

$\qquad\quad g_n \longrightarrow g \to S_2$

$L_2 -$

$\Rightarrow \quad f_n + g_n \longrightarrow h + g$

$\qquad\quad \downarrow$

$\qquad\quad f$

$\Rightarrow \quad f = h + g \in S_1 + S_2$