Lecture 36

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Measure and Integration

7/4/11

1

$$f : [a, b] \longrightarrow \mathbb{R}$$

f is monotonically increasing

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

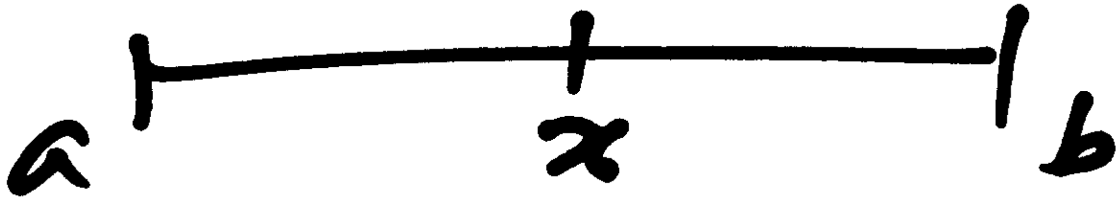
$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= f(x_n) - f(x_0)$$

$$= f(b) - f(a)$$

$$f : [a,b] \longrightarrow \mathbb{R}$$

$$V_a^b(f) < +\infty.$$



$$|f(x)| \le |f(x) - f(a)| + |f(a)|$$

$$\le V_a^b(P, f) + |f(a)|$$

$$\le \underline{V_a^b(f)} + \underline{|f(a)|}$$

$\Rightarrow f$ bounded.

$$f, g : [a,b] \longrightarrow \mathbb{R}$$

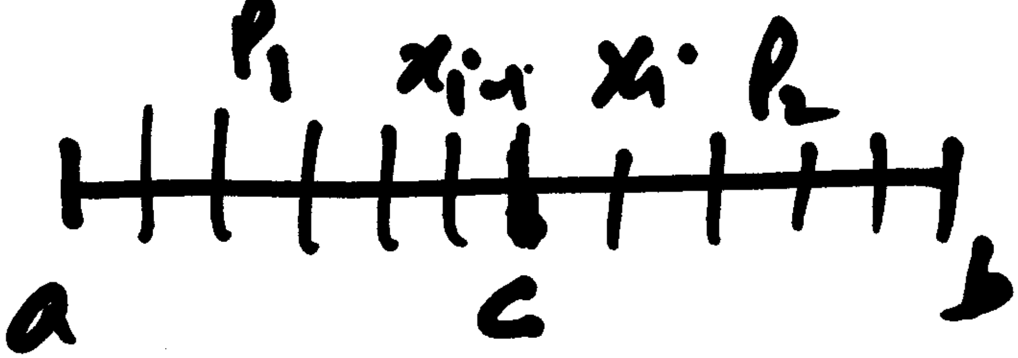
s.t $V_a^b(f) < +\infty$, $V_a^b(g) < +\infty$

P: a, x_1, x_2, $\ldots$, x_{n-1}, $b = x_n$, $\boxed{|f(x)| \le M \ \forall x}$

$$\sum_{i=1}^{n} \left| (fg)(x_i) - (fg)(x_{i-1}) \right|$$

$$\le M \sum_{i=1}^{n} \left| g(x_i) - g(x_{i-1}) \right|$$

$$V_a^b(P, fg) \le M\, V_a^b(g) \Rightarrow fg \text{ is of bdd variation}$$



4/5

$$V_a^b(f) = V_a^c(f) + V_c^b(f)$$

Let P of $[a,b]$, then

$$V_a^b(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$P = P_1 \cup P_2$, $P_1 - [a,c]$

$P_2 = [c,b]$

$$\le \cancel{\Sigma_i} \; \underline{V_a^c(P_1,f)} + \underline{V_c^b(P_2,f)}$$

$$V_a^b(P,f) \leq V_a^c(f) + V_c^b(f)$$

$$\Rightarrow \quad V_a^b(f) \leq V_a^c(f) + V_c^b(f)$$

Let $\varepsilon > 0$, select partitions P_1 of $[a,c]$ and P_2 of $[c,b]$

s.t.

$$V_a^c(f) - \varepsilon/2 \leq V_a^c(P_1,f)$$

$$V_a^b(f) - \varepsilon/2 \leq V_c^b(P_2,f)$$

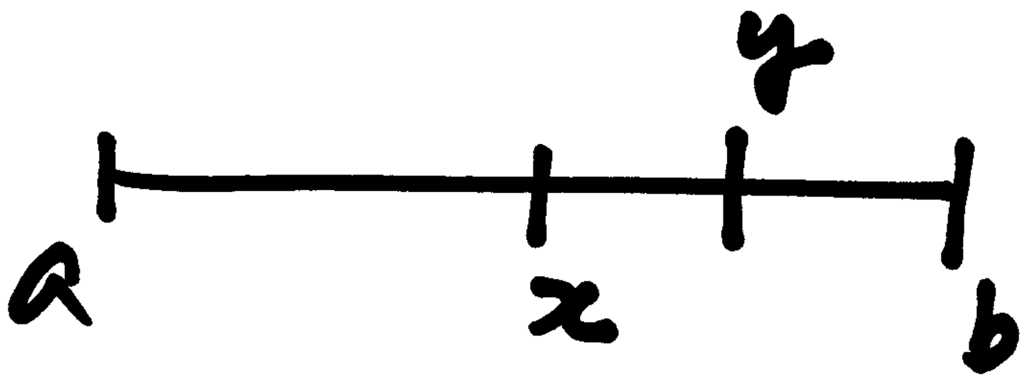
$$V_a^c(f) + V_c^b(f)$$

$$- \varepsilon \leq V_a^c(P_1, f) + V_c^b(P_2, f)$$

$$= V_a^b(P_1 \cup P_2, f)$$

$$\leq V_a^b(f)$$

$$\Rightarrow \quad V_a^c(f) + V_c^b(f) \leq V_a^b(f)$$



$V_a^x(f)$ is $\uparrow$

Let $x < y$

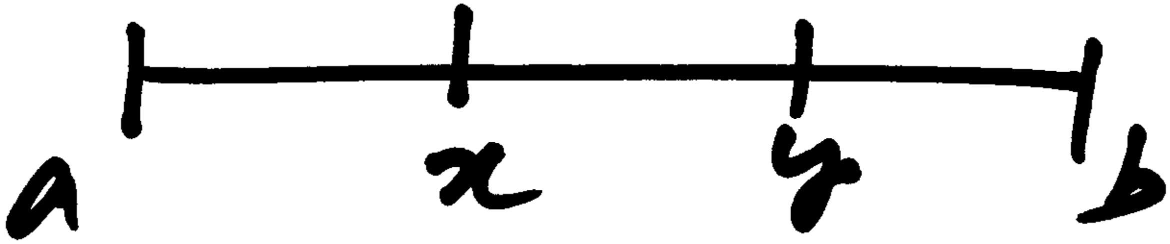
$\forall P$ of $[a,x]$, $P \cup \{y\} = P_1$ is a partition of $[a,y]$. Thus

$$V_a^x(P,f) \le V_a^x(P,f) + |f(y) - f(x)| = V_a^y(P_1,f) \le V_a^y(f)$$

$$\Rightarrow V_a^x(f) \le V_a^y(f)$$

$$f(y) - f(x) \leq V_a^y(f) - V_a^x(f)$$

$$\underline{V_a^y(f) - f(y)} \geq \underline{V_a^x(f) - f(x)}$$



$$f(y) - f(x) \leq |f(y) - f(x)|$$
$$\leq V_a^y(f) - V_a^x(f)$$

$$V_a^y(f) = V_a^x(f) + V_x^y(f)$$

$$\geq V_a^x(f)$$

$$V_a^x(f) \uparrow$$

$f \in L_1[a,b]$,

$$F(x) := \int_a^x f(t)\, d\lambda(t)$$

Then F is absolutely continuous.

<u>Case 1</u> f is bounded: $|f(x)| \le M$ $\forall\ x \in [a,b]$.

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

$$|F(x_i) - F(x_{i-1})| = \left| \int_a^{x_i} f\, d\lambda - \int_a^{x_{i-1}} f\, d\lambda \right|$$

$$|F(x_i)-F(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} f\,d\lambda\right|$$

$$\le \int_{x_{i-1}}^{x_i} |f|\,d\lambda$$

$$\Rightarrow \sum_{i=1}^{n} |F(x_i)-F(x_{i-1})| \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f|\,d\lambda$$

$$\le M \sum_{i=1}^{n} |x_i - x_{i-1}|$$

Choose $\delta > 0$ such that 1?

$$M\delta < \varepsilon$$

Then $\sum_{i=1}^{n} (b_i - a_i) < \delta$

$$\Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| \leq M \sum_{i=1}^{n} (b_i - a_i) < M\delta < \varepsilon$$

F is ab. cont if f is bdd.

In general $f \in L_1[a,b]$

Define $f_n = |f| \wedge n, \; n \geq 1$

Then

$f_n \in L_1, \quad f_n \leq |f|$

$f_n \longrightarrow |f|$

DCTh $\Longrightarrow$ $\int_a^b |f_n| \, d\lambda \longrightarrow \int_a^b |f| \, d\lambda .$

$\therefore \quad \int_E |f_n| \, d\lambda \longrightarrow \int_E |f| \, d\lambda$

$\forall \; E \subseteq [a,b].$

For $E \subseteq [a,b]$ fixed, we can find n_0 such that

$$\int_E |f|\, d\lambda - \int_E f_{n_0}\, d\lambda < \varepsilon$$

Then

$$\int_E \left(|f| - f_{n_0}\right) d\lambda < \varepsilon$$

$$\int_E |f|\, d\lambda = \int_E \left(|f| - f_{n_0}\right) d\lambda + \int_E f_{n_0}\, d\lambda$$

$$\leq \varepsilon + \int_E f_{n_0}\, d\lambda$$

$$\leq \varepsilon + n_0 \lambda(E)$$

If $n_0 \lambda(E) < \varepsilon$

Then

$$\int_E |f|\, d\lambda \leq 2\varepsilon$$

Let $\varepsilon > 0$ be given. Select $\delta > 0$

such that $n_0 \sum_{i=1}^{n} (b_i - a_i) < \varepsilon$

$$\Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)|$$

$$\leq \sum_{i=1}^{n} \int |f| \, d\lambda$$

$$= \int_E |f| \, d\lambda, \quad E = \bigcup (a_i, b_i)$$

$$< \varepsilon$$

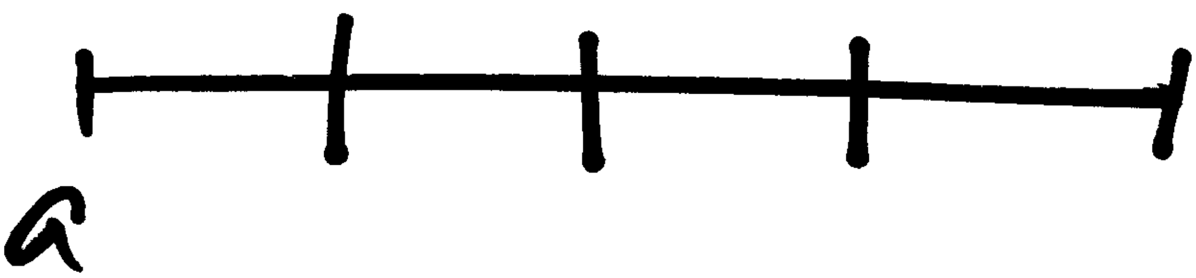
Thus $\forall\, \varepsilon > 0$, we can find $\delta > 0$ such that

(a_i, b_i), $i = 1, 2, \ldots, n$ disjoint in (a, b), $E = \bigcup_{i=1}^{n} (a_i, b_i)$

if $n_0 \underline{\sum_{i=1}^{n} (b_i - a_i)} = \quad < \varepsilon$

$= n_0 \delta <$

$$\Rightarrow \int_E |f| \, d\lambda < \varepsilon$$



Let $P = \{a = x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ such that $(x_i - x_{i-1}) < \delta$. Then

$$V_{x_{i-1}}^{x_i}(f) < 1$$

$$\implies V_a^b(f) = \sum_{i=1}^{n} V_{x_{i-1}}^{x_i}(f) \le n < +\infty$$

$f : [a,b] \longrightarrow \mathbb{R}$

f is ab. cont

$\Rightarrow$ f is of bdd variation

Let $\varepsilon = 1$ be given. Select

$\delta > 0$ st (a_i, b_i), $i = 1, \ldots, n$

disjoint in (a,b) with

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)| < 1$$