Lecture 38

Measure and Integration

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5/5/11

ν, μ on $(X, \mathcal{S})$

$\nu \ll \mu$, ν is finite

<u>To Show</u> $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t

$$\mu(E) < \delta \implies \nu(E) < \varepsilon$$

Pf Suppose not. Then ~~$\forall \varepsilon$~~

$\exists\ \varepsilon > 0$ s.t $\forall\ \delta$, $\exists\ E \in \mathcal{S}$

s.t $\mu(E) < \delta$ but $\nu(E) \geq \varepsilon$

Apply this for $\delta = \frac{1}{2^n}$.

$\forall n$, $\exists$ a set $E_n \in \mathcal{S}$ s.t

$\mu(E_n) < \frac{1}{2^n}$ but $\nu(E_n) \geq \varepsilon$

Define

$$A_n = \bigcup_{k=n}^{\infty} E_k, \quad A = \bigcap_{n=1}^{\infty} A_n.$$

<u>Note</u> $\mu(A_n) \leq \sum_{k=n}^{\infty} \mu(E_k) = \frac{1}{2^{n+1}}$

and $A_n \downarrow A$. Thus, ν finite

$$= \nu(A) = \lim_{n\to\infty} \downarrow \nu(A_n).$$

And

$$\nu(A_n) \geq \nu(E_n) \geq \varepsilon$$

$$\Rightarrow \quad \nu(A) \geq \varepsilon.$$

But

$$\mu(A_n) \leq \frac{1}{2^{n+1}}$$

$$\Rightarrow \quad \mu(A) \leq \mu(A_n) \quad \forall n$$

$$\leq \frac{1}{2^{n+1}} \quad \forall n$$

$$\Rightarrow \quad \mu(A) = 0 \qquad \otimes$$

<u>Converse</u> $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$

s.t. $\mu(E) < \delta \Rightarrow \nu(E) < \varepsilon$ ||

<u>Claim</u> $\nu \ll \mu$?

Let $\mu(E) = 0$. Then $\forall\, \varepsilon, \forall\, \delta$

$$\mu(E) = 0 < \delta$$

$$\Longrightarrow \underline{\nu(E) < \varepsilon}.$$

$$\Longrightarrow \nu(E) = 0.$$

__Assume__ $F : \mathbb{R} \xrightarrow[\text{left cont}]{\text{m.i}} \mathbb{R}$

$$\mu_F(a,b] := F(b) - F(a)$$

__Let $\mu_F \ll \lambda$.__ __To Show__ $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every interval $(a,b]$?

To show $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ such that $[a_i, b_i]$ are finite disjoint intervals in $(a,b]$ with

$$\sum_{i=1}^{n}(b_i-a_i)<\delta \Rightarrow \sum_{i=1}^{n}|F(b_i)-F(a_i)| < \varepsilon \,?$$

<u>Let $\varepsilon>0$ be given</u>

Since $\mu_F \ll \lambda$,

$\exists\ \delta>0$ such that $\forall\ E\in\mathcal{B}_{[a,b]}$

$$\lambda(E)<\delta \Rightarrow \mu_F(E)<\varepsilon.$$

In particular, if

$$E=\bigsqcup_{i=1}^{n}[a_i,b_i] \quad \text{s.t.}$$

s.t $\lambda(E) = \sum_{i=1}^{n} (b_i - a_i) < \delta$

$$\Longrightarrow \mu_F(E) = \sum_{i=1}^{n} (F(b_i) - F(a_i)) < \varepsilon$$

Hence $F : [a,b] \longrightarrow \mathbb{R}$

is absolutely continuous.

~~Assume~~ $\cancel{\mu_F \ll \lambda}$

~~To show~~

Assume. $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous.

To show

$$\mu_F \ll \lambda?$$

Let $E \in \mathcal{B}_{\mathbb{R}}$, $\lambda(E) = 0$.

To show $\mu_F(E) = 0$?

Enough to show that

$$\mu_F(E \cap [a,b]) = 0$$

$\forall$ interval $[a,b]$.

<u>Since</u> $\lambda(E \cap [a,b]) = 0$.

$*$ We can find, for a given <u>$\varepsilon > 0$</u>,
intervals $(a_n, b_n]$, $n = 1, 2, \ldots$
such that

$$E \cap [a,b] \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n]$$

and $$\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$$

<u>Now</u> let $\alpha > 0$. Then by absolute continuity of F, $\exists\ \delta > 0$ such that whenever intervals $[a_n, b_n]$ are disjoint in $[a, b]$, with

$$\sum_{n=1}^{k} (b_n - a_n) < \delta \implies \sum_{n=1}^{k} F(b_n) - F(a_n) < \alpha$$

By (*) with $\varepsilon = \delta$, we have

$$\sum_{n=1}^{k} \left(F(b_n) - F(a_n)\right) < \alpha$$

$$\forall\ k$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[F(b_n) - F(a_n)\right] < \alpha$$

$$\Rightarrow \mu_F(E \cap [a,b])$$

$$\leq \sum_{n=1}^{\infty} \mu_F((a_n, b_n])$$

$$= \sum_{n=1}^{\infty} \left[F(b_n) - F(a_n)\right] < \alpha$$

$$\Rightarrow \mu_F(E \cap [a,b]) = 0$$

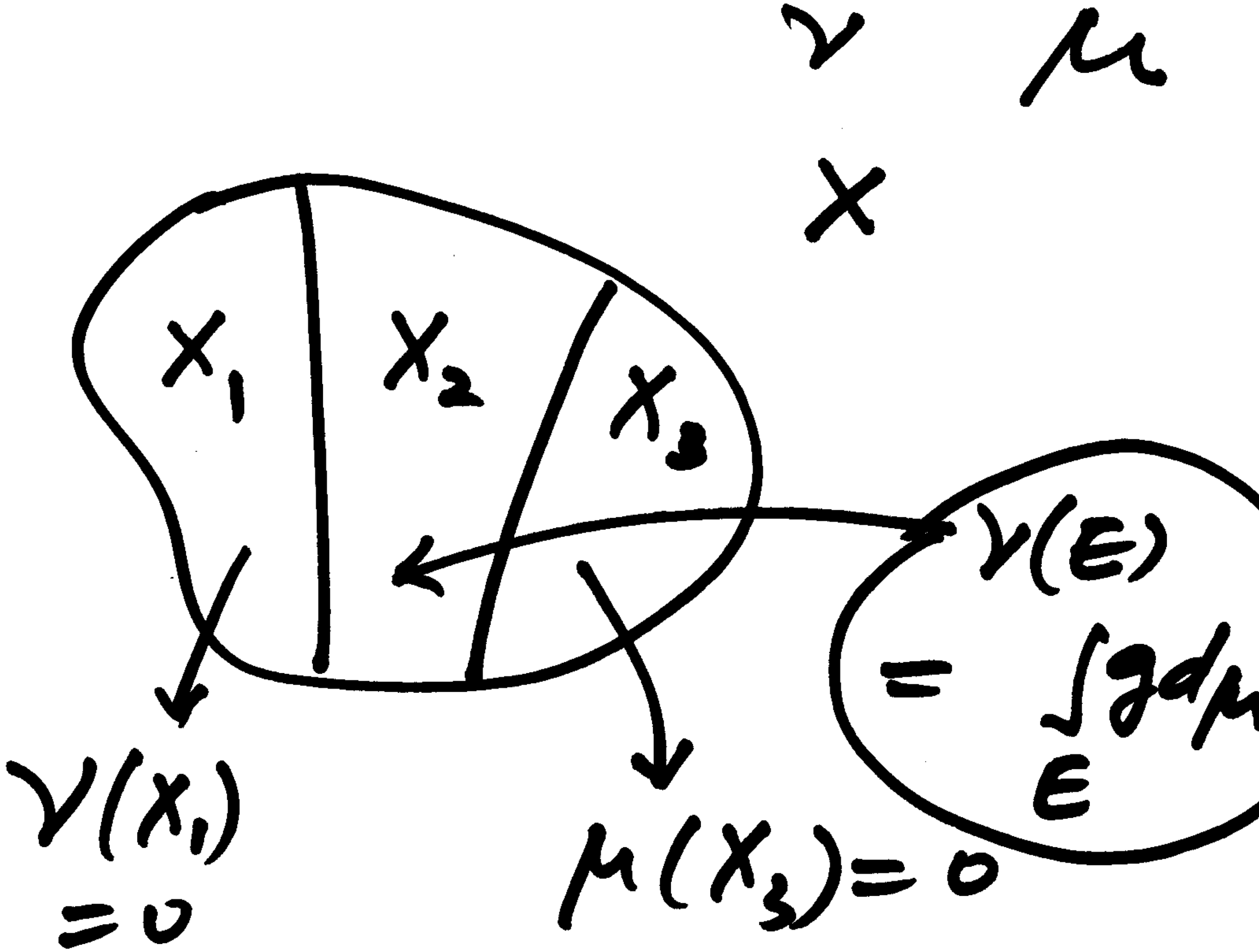
Hence $\lambda(E) = 0$

$$\Rightarrow \mu_F(E \cap [a,b]) = 0$$

$$\forall\ a,b$$

$$\Rightarrow \mu_F(E) = 0$$

Hence $\mu_F \ll \lambda$.


ν μ
X
X_1 X_2 X_3
$\nu(E) = \int_E g\,d\mu$
$\nu(X_1) = 0$
$\mu(X_3) = 0$