# Classes of sets (Lectures 1, 2, 3 and 4)

### 1.1. Semi-algebra and algebra of sets

(1.1) Let $\mathcal{F}$ be any collection of subsets of a set $X$. Show that $\mathcal{F}$ is an algebra if and only if the following hold:
- (i) $\emptyset, X \in \mathcal{F}$.
- (ii) $A^c \in \mathcal{F}$ whenever $A \in \mathcal{F}$.
- (iii) $A \cup B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$.

(1.2) Let $\mathcal{F}$ be an algebra of subsets of $X$. Show that
- (i) If $A, B \in \mathcal{F}$ then $A \triangle B := (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$.
- (ii) If $E_1, E_2, \ldots, E_n \in \mathcal{F}$ then there exists sets $F_1, F_2, \ldots, F_n \in \mathcal{F}$ such that $F_i \subseteq E_i$ for each $i$, $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} F_j$.

The next set of exercise describes some methods of constructing algebras and semi-algebras.

(1.3) Let $X$ be a nonempty set. Let $\emptyset \neq E \subseteq X$ and let $\mathcal{C}$ be a semi-algebra (algebra) of subsets of $X$.

$$\mathcal{C} \cap E := \{A \cap E \,|\, A \in \mathcal{C}\}.$$

Note that $\mathcal{C} \cap E$ is the collection of those subsets of $E$ which are elements of $\mathcal{C}$. Show that $\mathcal{C} \cap E$ is a semi-algebra (algebra) of subsets of $E$.

(1.4) Let $X, Y$ be two nonempty sets and $f : X \longrightarrow Y$ be any map. For $E \subseteq Y$, we write $f^{-1}(E) := \{x \in X \,|\, f(x) \in E\}$. Let $\mathcal{C}$ be any semi-algebra (algebra) of subsets of $Y$. Show that

$$f^{-1}(\mathcal{C}) := \{f^{-1}(E) \mid E \in \mathcal{C}\}$$

is a semi-algebra (algebra) of subsets of $X$.

(1.5) Give examples of two nonempty sets $X, Y$ and algebras $\mathcal{F}, \mathcal{G}$ of subsets of $X$ and $Y$, respectively such that $\mathcal{F} \times \mathcal{G} := \{A \times B \,|\, A \in \mathcal{F}, B \in \mathcal{G}\}$ is not an algebra. (It will of course be a semi-algebra.

(1.6) Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a family of algebras of subsets of a set $X$. Let $\mathcal{F} := \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Show that $\mathcal{F}$ is an algebra of subsets of $X$. Is $\mathcal{F}$ a semi-algebra of subsets of $X$ If each $\mathcal{F}_\alpha$ a semi-algebra?

(1.7) Let $\{\mathcal{F}_n\}_{n \geq 1}$ be a sequence of algebras of subsets of a set $X$ and $\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$. In general, $\mathcal{F}$ is not an algebra. Under what conditions on $\mathcal{F}_n$ can you conclude that is also an algebra?

(1.8) *Let $\mathcal{C}$ be any collection of subsets of a set $X$. Then there exists a unique algebra $\mathcal{F}$ of subsets of $X$ such that $\mathcal{C} \subseteq \mathcal{F}$ and if $\mathcal{A}$ is any other algebra such that $\mathcal{C} \subseteq \mathcal{A}$, then $\mathcal{F} \subseteq \mathcal{A}$.* This unique algebra is called the **algebra generated by** $\mathcal{C}$ and is denoted by $\mathcal{F}(\mathcal{C})$.

(1.9) Show that he algebra generated by $\mathcal{I}$, the class of all intervals, is $\{E \subseteq \mathbb{R} \mid E = \bigcup_{k=1}^{n} I_k, I_k \in I,\, I_k \cap I_\ell = \emptyset \text{ if } 1 \leq k \neq \ell \leq n\}$.

(1.10) Let $\mathcal{C}$ be any semi-algebra of subsets of a set $X$. Show that $\mathcal{F}(\mathcal{C})$, the algebra generated by $\mathcal{C}$, is given by

$$\{E \subseteq X \mid E = \bigcup_{i=1}^{n} C_i, C_i \in \mathcal{C} \text{ and } C_i \cap C_j = \emptyset \text{ for } i \neq j, n \in \mathbb{N}\}.$$

This gives a description of $\mathcal{F}(\mathcal{C})$, the algebra generated by a semi-algebra $\mathcal{C}$. In general, no description is possible for $\mathcal{F}(\mathcal{C})$ when $\mathcal{C}$ is not a semi-algebra.

(1.11) Let $X$ be any nonempty set and $\mathcal{C} = \{\{x\} \mid x \in X\} \bigcup \{\emptyset, X\}$. Is $\mathcal{C}$ a semi-algebra of subsets of $X$? What is the algebra generated by $\mathcal{C}$? Does your answer depend upon whether $X$ is finite or not?

(1.12) Let $\mathcal{C}$ be any collection of subsets of a set $X$ and let $E \subseteq X$. Let

$$\mathcal{C} \cap E := \{C \cap E \mid C \in \mathcal{C}\}.$$

Then the following hold:

(a)

$$\mathcal{C} \cap E \subseteq \mathcal{F}(\mathcal{C}) \cap E := \{A \cap E \mid A \in \mathcal{F}(\mathcal{C})\}.$$

Deduce that

$$\mathcal{F}(\mathcal{C} \cap E) \subseteq \mathcal{F}(\mathcal{C}) \cap E.$$

(b) Let

$$\mathcal{A} = \{A \subseteq X \mid A \cap E \in \mathcal{F}(\mathcal{C} \cap E)\}.$$

Then, $\mathcal{A}$ is an algebra of subsets of $X$, $\mathcal{C} \subseteq \mathcal{A}$ and

$$\mathcal{A} \cap E = \mathcal{F}(\mathcal{C} \cap E).$$

(c) Using (a) and (b), deduce that $\mathcal{F}(\mathcal{C}) \cap E = \mathcal{F}(\mathcal{C} \cap E)$.

**Optional Exercises**

(1.13) Let $\mathcal{C}$ be a semi-algebra of subsets of a set $X$. A set $A \subseteq X$ is called a $\sigma$**-set** if there exist sets $C_i \in \mathcal{C}, i = 1, 2, \ldots$, such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} C_i = A$. Prove the following:

(i) For any finite number of sets $C, C_1, C_2, \ldots, C_n$ in $\mathcal{C}$, $C \setminus (\bigcup_{i=1}^{n} C_i)$ is a finite union of pairwise disjoint sets from $\mathcal{C}$ and hence is a $\sigma$-set.

(ii) For any sequence $\{C_n\}_{n\geq 1}$ of sets in $\mathcal{C}$, $\bigcup_{n=1}^{\infty} C_n$ is a $\sigma$-set.

(iii) A finite intersection and a countable union of $\sigma$-sets is a $\sigma$-set.

(1.14) Let $Y$ be any nonempty set and let $X$ be the set of all sequences with elements from $Y$, i.e.,

$$X = \{\underline{x} = \{x_n\}_{n\geq 1} \mid x_n \in Y, n = 1, 2, \ldots\}.$$

For any positive integer $k$ let $A \subseteq Y^k$, the $k$-fold Cartesian product of $Y$ with itself, and let $i_1 < i_2 < \cdots < i_k$ be positive integers. Let

$$C(i_1, i_2, \ldots, i_k; A) := \{\underline{x} = (x_n)_{n\geq 1} \in X \mid (x_{i_1}, \ldots, x_{i_k}) \in A\}.$$

We call $C(i_1, i_2, \ldots, i_k; A)$ a $k$-**dimensional cylinder set** in $X$ with base $A$. Prove the following assertions:

(a) Every $k$-dimensional cylinder can be regarded as a $n$-dimensional cylinder also for $n \geq k$.

(b) Let

$$\mathcal{A} = \{E \subset X \mid E \text{ is an } n\text{-dimensional cylinder set for some } n\}.$$

Then, $\mathcal{A} \cup \{\emptyset, X\}$ is an algebra of subsets of $X$.

## 1.2. Sigma algebra and monotone class

(1.15) Let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $X$ and let $Y \subseteq X$. Show that $\mathcal{S} \cap Y := \{E \cap Y \mid E \in \mathcal{S}\}$ is a $\sigma$-algebra of subsets of $Y$.

(1.16) Let $f : X \to Y$ be a function and $\mathcal{C}$ a nonempty family of subsets of $Y$. Let $f^{-1}(\mathcal{C}) := \{f^{-1}(C) \mid C \in \mathcal{C}\}$. Show that $\mathcal{S}(f^{-1}(\mathcal{C})) = f^{-1}(\mathcal{S}(\mathcal{C}))$.

(1.17) Let $X$ be an uncountable set and $\mathcal{C} = \{\{x\} \mid x \in X\}$. Identify the $\sigma$-algebra generated by $\mathcal{C}$.

(1.18) Let $\mathcal{C}$ be any class of subsets of a set $X$ and let $Y \subseteq X$. Let $\mathcal{A}(\mathcal{C})$ be the algebra generated by $\mathcal{C}$.

(i) Show that $\mathcal{S}(\mathcal{C}) = \mathcal{S}(\mathcal{A}(\mathcal{C}))$.

(ii) Let $\mathcal{C} \cap Y := \{E \cap Y \mid E \in \mathcal{C}\}$. Show that $\mathcal{S}(\mathcal{C} \cap Y) \subseteq \mathcal{S}(\mathcal{C}) \cap Y$.

(iii) Let

$$\mathcal{S} := \{E \cup (B \cap Y^c) \mid E \in \mathcal{S}(\mathcal{C} \cap Y), B \in \mathcal{S}(C)\}.$$

Show that $\mathcal{S}$ is a $\sigma$-algebra of subsets of $X$ such that $\mathcal{C} \subseteq \mathcal{S}$ and $\mathcal{S} \cap Y = \mathcal{S}(\mathcal{C} \cap Y)$.

(iv) Using (i), (ii) and (iii), conclude that $\mathcal{S}(\mathcal{C} \cap Y) = \mathcal{S}(\mathcal{C}) \cap Y$.

(1.19) Let $X$ be any topological space. Let $\mathcal{U}$ denote the class of all open subsets of $X$ and $\mathcal{C}$ denote the class of the all closed subsets of $X$.

(i) Show that
$$\mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{C}).$$
This is called the $\sigma$-algebra of **Borel subsets** of $X$ and is denoted by $\mathcal{B}_X$.

(ii) Let $X = \mathbb{R}$. Let $\mathcal{I}$ be the class of all intervals and $\tilde{\mathcal{I}}$ the class of all left-open right-closed intervals. Show that $\mathcal{I} \subset \mathcal{S}(\mathcal{U})$, $\mathcal{I} \subset \mathcal{S}(\tilde{\mathcal{I}})$, $\tilde{\mathcal{I}} \subset \mathcal{S}(\mathcal{I})$ and hence deduce that
$$\mathcal{S}(\mathcal{I}) = \mathcal{S}(\tilde{\mathcal{I}}) = \mathcal{B}_{\mathbb{R}}.$$

(1.20) Prove the following statements:

(i) Let $\mathcal{I}_r$ denote the class of all open intervals of $\mathbb{R}$ with rational endpoints. Show that $\mathcal{S}(\mathcal{I}_r) = \mathcal{B}_{\mathbb{R}}$.

(ii) Let $\mathcal{I}_d$ denote the class of all subintervals of $[0,1]$ with dyadic endpoints (i.e., points of the form $m/2^n$ for some integers $m$ and $n$). Show that $\mathcal{S}(\mathcal{I}_d) = \mathcal{B}_{\mathbb{R}} \cap [0,1]$.

(1.21) Let $\mathcal{C}$ be any class of subsets of $X$. Prove the following:

(i) If $\mathcal{C}$ is an algebra which is also a monotone class, show that $\mathcal{C}$ is a $\sigma$-algebra.

(ii) $\mathcal{C} \subseteq \mathcal{M}(\mathcal{C}) \subseteq \mathcal{S}(\mathcal{C})$.

(1.22) (**$\sigma$-algebra monotone class theorem** ) *Let $\mathcal{A}$ be an algebra of subsets of a set $X$. Then , $\mathcal{S}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.*

Prove the above statement by proving the following:

(i) $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{S}(\mathcal{A})$.

(ii) Show that $\mathcal{M}(\mathcal{A})$ is closed under complements by proving that for
$$\mathcal{B} := \{E \subseteq X \mid E^c \in \mathcal{M}(\mathcal{A})\},$$
$\mathcal{A} \subseteq \mathcal{B}$, and $\mathcal{B}$ is a monotone class. Hence deuce that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{B}$.

(iii) For $F \in \mathcal{M}(\mathcal{A})$, let
$$\mathcal{L}(F) := \{A \subseteq X \mid A \cup F \in \mathcal{M}(\mathcal{A})\}.$$
Show that $E \in \mathcal{L}(F)$ iff $F \in \mathcal{L}(E)$, $\mathcal{L}(F)$ is a monotone class, and $\mathcal{A} \subseteq \mathcal{L}(F)$ whenever $F \in \mathcal{A}$. Hence, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{L}(F)$, for $F \in \mathcal{A}$.

(iv) Using (iii), deduce that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{L}(E)$ for every $E \in \mathcal{M}(\mathcal{A})$,i.e., $\mathcal{M}(\mathcal{A})$ is closed under unions also. Now use exercise (1.22) to deduce that $\mathcal{S}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.