Probability and Statistics
Test Set 6

1. Let (X,Y) have the joint pdf $f(x,y) = e^{-(x+y)}$, $x > 0$, $y > 0$. Check whether X and Y are independent? Find $P(1 < X - Y < 2)$, $P(X < 2Y \mid X < 3Y)$, $P(1 < Y < 2 \mid X = 2)$. Also find c such that $P(X - Y < c) = 0.25$.

2. Let (X,Y) have the joint pdf $f(x,y) = x + y$, $0 < x < 1$, $0 < y < 1$. Are X and Y independent? Find V(X+Y), Corr(X,Y), $V(Y| X=x)$ and $V(X \mid Y = y)$.

3. Let (X,Y) have the joint probability density function

   $f_{X,Y}(x, y) = (2 - x - y)$, $0 < x < 1$, $0 < y < 1$.

   Find $E(Y \mid X=x)$, $Var(Y \mid X=x)$, Cov(X,Y), $Var(X \mid Y=y)$ and Corr(X, Y).

4. The age of siblings at a certain community school can be approximated by a BVN(11, 13, 16, 16, 0.8) (the first variable corresponds to the younger child and the second to his/her sibling). Find
   a. the proportion of younger siblings below 10 years.
   b. The proportion of elder siblings over 15 having younger sibling of age 12.

5. The life of a equipment (X) and the width of a component in it (Y) are distributed as a bivariate normal with $\mu_1 = 2$ years, $\mu_2 = 0.2$ cm, $\sigma_1 = 0.5$, $\sigma_2 = 0.05$, $\rho = 0.85$. If the width of the component is 0.15 cm, what is the probability that the equipment will last 2.5 years?

6. Let (X, Y) be continuous with the joint probability density function given by

   $$f_{X,Y}(x, y) = \frac{1}{y} e^{-(y + x/y)}, \; x > 0, y > 0.$$

   Find the marginal density of Y and the conditional density of X given Y = y. Hence, or otherwise, calculate, E(Y), E(X), Var(Y), Var(X), Cov(X, Y) and Corr(X, Y).

7. In a probability class the marks of students in two consecutive tests are seen to follow a bivariate normal distribution with $\mu_1 = 50$, $\mu_2 = 70$, $\sigma_1 = 4$, $\sigma_2 = 5$ and $\rho = 0.9$. If a student scores 60 on the first test, what is the probability that he/she will score more than 80 in the second one? If $\rho = -0.9$, what will be this probability?

8. The average temperatures recorded (in Celsius) at a weather station in December is a random variable X and the average temperatures recorded at the same station in January is a random variable Y. Let (X, Y) have a bivariate normal distribution with parameters $\mu_1 = 4$, $\mu_2 = 2$, $\sigma_1 = 0.8$, $\sigma_2 = 0.6$, $\rho = 0.7$. Find $P(X \leq 3)$, $P(Y \leq 1 \mid X = 3)$ and $E(X \mid Y = 1)$.

9. Let X and Y have joint density function

$$f(x,y) = 1 - \alpha(1-2x)(1-2y), \quad 0 < x < 1, 0 < y < 1;$$
$$= 0, \qquad \text{otherwise.}$$

Find correlation between X and Y and hence determine the valid range of α. Show that X and Y are independent if and only if they are uncorrelated.

10. Let X denote the number of functions and Y denote the maximum number of lines (in 100s) in a 'C' program by an expert programmer for an organization. Assume that the joint probability mass function of (X,Y) is given by

X\ Y	1	2	3	4
0	0.05	0.10	0.05	0.01
1	0.09	0.12	0.08	0.03
2	0.06	0.10	0.10	0.01
3	0.05	0.07	0.06	0.02

a. Find the probability that a randomly selected programme contains at most two functions and has at most 100 lines.
b. Find marginal probability mass functions of X and Y. Calculate $\rho_{X,Y}$.
c. Find the probability that a randomly selected programme contains at least two functions given that it contains at least one hundred lines.

11. Let X denote the purchase value of a share and Y the value at sale for an expert investor (the maximum values are scaled to unity). Assume that the joint density of X and Y is given by

$$f(x, y) = k\,x\,y \quad , \text{ if } 0 < 2y < x < 1$$
$$= 0 \quad , \text{ elsewhere.}$$

(a) Find the value of k.
(b) Find the probability that the investor is able to sell a randomly selected share at a price four times more than the purchase price.
(c) Find the expected sale price if the purchase price is below 1/2.
(d) Find the probability that purchase price is below ¼ and the sale price is more than ¾ .
(e) Find the conditional distribution of X given Y = ½ .
(f) Find coefficient of correlation between X and Y.

12. Let (X, Y) be continuous with the joint probability density function

$$f(x,y) = \frac{1}{4}(1+xy), \quad |x| < 1, |y| < 1$$
$$= 0, \qquad \text{otherwise.}$$

Find P(2X < Y), P(|X+Y| < ½) and $\rho_{X,Y}$