Probability and Statistics
Hints/Solutions to Test Set 3

1. (a) No, as it is not continuous from right at x =2/3..
   (b) Yes
   (c) Yes
   (d) No

2. $P\left(\frac{1}{2} \le X \le \frac{5}{2}\right) = F\left(\frac{5}{2}\right) - F\left(\frac{1}{2}-\right) = \frac{3}{4} - \frac{1}{16} = \frac{11}{16}.$

   $P(1 < X < 3) = F(3-) - F(1) = \frac{7}{8} - \frac{1}{4} = \frac{5}{8}.$

   The random variable X is continuous in the intervals (0, 1), (1, 2) and (2, 3) with the uniform densities1/8, 1/8 and 1/4, and discrete at points 1, 2 and 3 with probabilities 1/8, 1/4 and 1/8 respectively. So

   $E(X) = \int_0^1 \frac{x}{8}\,dx + \int_1^2 \frac{x}{8}\,dx + \int_2^3 \frac{x}{4}\,dx + 1.\frac{1}{8} + 2.\frac{1}{4} + 3.\frac{1}{8} = \frac{15}{8}.$

   $E(X^2) = \int_0^1 \frac{x^2}{8}dx + \int_1^2 \frac{x^2}{8}dx + \int_2^3 \frac{x^2}{4}dx + 1^2.\frac{1}{8} + 2^2.\frac{1}{4} + 3^2.\frac{1}{8} = \frac{25}{6}.$

   $V(X) = \frac{125}{192}.$ Median (X) = 2.

3. $X \sim Bin(3, \frac{1}{2}), \quad Y \sim P(\lambda).$

   So P(X + Y = 1)
   = P(X = 0, Y = 1) + P(X = 1, Y = 0)
   = P(X = 0) P(Y = 1) + P(X = 1) P(Y = 0), as X and Y are independent
   $= \left(\frac{1}{2}\right)^3 .e^{-1} + 3.\left(\frac{1}{2}\right)^3 .e^{-1} = \frac{e^{-1}}{2} = 0.1839.$

4. The cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{8}, & 0 \le x < 1, \\ \frac{6x-5}{8}, & 1 \le x < 2, \\ 1 - \frac{(3-x)^2}{8}, & 2 \le x < 3, \\ 1, & \text{if } x \ge 3. \end{cases}$$

   E(X) = 3/2, Median (X) = 3/2, Var(X) = 1/4.

5. Let X = the number of organisms at the second stage,
   Let Y = the number of organisms at the third stage
   Then X → 1, 2, 3; Y →1, 2, …, 9.
   P(Y = 1) = 1/9, P(Y = 2) = 4/27, P(Y = 3) = 16/81, P(Y = 4) = 4/27 = P(Y = 5),
   P(Y = 6) = 10/81, P(Y = 7) = 2/27, P(Y = 8) = 1/27, P(Y = 9) = 1/81.

6. Let X be the number of tests required. Then X is either 1 or 11.
   $P(X=1) = P(\text{ none has the disease}) = (0.99)^{10}$,
   $P(X = 11) = P(\text{ at least one has disease}) = 1 - (0.99)^{10}$,
   $E(X) = 11 - 10(0.99)^{10}$.

7. $0 \le p_X(i) \le 1, \quad i = 1, \cdots, 4$ yields $-\frac{1}{4} \le d \le \frac{1}{3}$.

   $E(X) = \frac{9d+2}{4}, E(X^2) = \frac{11d+6}{4}, V(X) = \frac{20+8d-81d^2}{16}$.

   V(X) is minimized at $d = -\frac{1}{4}$.

8. $\sum_{x=0}^{\infty} p_X(x) = k$, so k = 1.

$$
\begin{aligned}
F(x) &= 0, && \text{if} \quad x < 0, \\
&= 1/2, && \text{if} \quad 0 \le x < 1, \\
&= 2/3, && \text{if} \quad 1 \le x < 2, \\
&\vdots \\
&= n/(n+1), && \text{if} \quad n-1 \le x < n, \\
&\vdots
\end{aligned}
$$

   E(X) does not exist. Any M between 0 and 1 is a median.

9. Let X denote the marks on a test. Then using Chebyshev's inequality, we get

   $P(X < 60 \text{ or } X > 90) = P(|X - 75| > 15) \le \frac{V(X)}{(15)^2} = \frac{1}{25}$.