Probability and Statistics
Hints/Solutions to Test Set 4

1. $\text{P(Ruby wins)} = \dfrac{\binom{6}{2}\binom{3}{1} + \binom{6}{3}}{\binom{9}{3}} = \dfrac{65}{84}.$

2. Suppose n missiles are fired and X is the number of successful hits. Then $X \sim \text{Bin}(n, 0.75)$. We want n such that $P(X \geq 3) \geq 0.95$, or $P(X = 0) + P(X = 1) + P(X = 3) \leq 0.05$. This is equivalent to $10(9n^2 - 3n + 2) \leq 4^n$. The smallest value of n for which this is satisfied is n =6.

3. Let P_n denote the probability of an n-component system to operate effectively. Then $P_3 = 1-(1-p)^3$ and $P_6 = 1-(1-p)^6 - \binom{6}{1} p(1-p)^5$. Now

$$P_6 - P_3 = -p(1-p)^3(5p^2 - 9p + 3),$$

which is > 0, if $p > \dfrac{9-\sqrt{21}}{10} \simeq 0.4417.$

4.
$$\begin{aligned} \text{P(returning a pack)} &= P(X > 1) = 1 - P(X=0) - P(X=1) \\ &= 1 - (0.99)^{10} - 10(0.99)^9(0.01), \text{say.} \end{aligned}$$

Let Y denote the number of packs returned. Then $Y \sim \text{Bin}(3, p)$.

$P(Y = 0) + P(Y = 1) = 0.9999455.$

5.
$$\begin{aligned} P(Y = 1000j) &= \frac{e^{-5}5^j}{j!}, \qquad j = 0, 1, \cdots, 9, \\ &= \sum_{j=10}^{\infty} \frac{e^{-5}5^j}{j!}, \quad j = 10. \end{aligned}$$

6. Let A be the event that person gets a viral infection and B denote the event that the he/she follows the new health regime. Let X be the number of times an individual contracts the viral infection in a year. Then $X \mid B \sim P(1)$, and $X \mid B^C \sim P(5)$.

$P(A^C \mid B) = P(X = 0 \mid B) = e^{-1}$. $P(A^C \mid B^C) = P(X = 0 \mid B^C) = e^{-5}$.

Using Bayes Theorem $P(B \mid A^C) = \dfrac{3}{3 + e^{-4}} = 0.9939.$

7. Let X be the number of defects. Then $X \sim P(300)$. Ley Y be the number of defects in 5% of the pages. Then $Y \sim P(3)$. The required probability is $P(Y \leq 2) = 0.4232$.
8. Use geometrical probability.

9. Let X denote the life in months of a machine. Then $P(X > 200) = e^{-2}$. Let Y denote the number of machines working after 200 months. Then $Y \sim \text{Bin}(20, e^{-2})$. So $P(Y \geq 2) = 07746$.

10. $P(X > 6) = P(X > 6 \mid I)P(I) + P(X > 6 \mid II)P(II) = e^{-1}.\frac{1}{5} + e^{-3}.\frac{4}{5}.$

11 . $P(\text{system fails before time t}) = 1 - \exp\{-t\sum_{i=1}^{n} 2^{-i}\}.$

P(only component j fails before time t| system fails before time t)

$$= \frac{(1\text{-}\exp\{-2^{-j}t\})\exp\{-t\sum_{\substack{i=1\\ i\neq j}}^{n} 2^{-i}\}}{1 - \exp\{-t\sum_{i=1}^{n} 2^{-i}\}}.$$

12. Here, $r = 4$, $\lambda = 1/5$, Reqd Prob = 0.3528.

13. $P(X \le 6 \mid X \ge 5) = 0.9$ gives $\alpha = 0.0253$.

$P(X > 8) = \exp\{-0.0253 \times 8^2\} = 0.198.$

14. μ = 58.13, σ = 10.26, Percentage of students getting grade 'B' is 37.86, Percentage of students getting grade 'C' is 47.14.

15. Let X denote the diameter (in cm). Then X ~ $N(3, 0.005^2)$.

$P(\text{ chalk piece is defective}) = 1 - P(2.99 < X < 3.01)$

$= 2\Phi(-2) = 0.0455.$

16. $P(4.95 < X < 5.05) = P(-1.66 < Z < 1.66) = 2\Phi(1.66) - 1 = 0.903.$

So the percentage of defectives = 100 *0.097 = 9.7%.

When $X \sim N(5.0, \sigma^2)$, then $P(4.95 < X < 5.05) \ge 0.99$ is equivalent to

$\Phi(0.05/\sigma) \ge 0.995$, or $\sigma \le 0.00194$.

17. Let X denote the distance (in cm.) that athlete jumps.

Then X ~ N(800, 100). Let c be such that P(X > c) =0.95.

Then (800 – c) / 10 = 1.645 and so c = 783.55 cm.

Further let d be such that P(X > d) = 0.1.

Then (800 – d)/10 = –1. 28 and so d = 812.80 cm.

18. Let X denote the marks. Then X ~ N(74, 62.41).

Ans. (a) 64 (b) 86 (c) 77

19. ln Y ~ N(0.8, 0.01). So $P(Y > 2.7) = P(\ln Y > 0.9933) = P(Z > 1.93) = 0.0268.$

Let c be such that $P(0.8 - c < \ln Y < 0.8 + c) = 0.95.$ This is equivalent to

$P(-\frac{c}{0.1} < Z < \frac{c}{0.1}) = 0.95$, or $\Phi\left(\frac{c}{0.1}\right) = 0.975$, so $c = 0.196$. Therefore

$P(1.8294 < Y < 2.7074) = 0.95.$