Probability and Statistics
Hints/Solutions to Test Set 6

1. Here note that X and Y are independent. The probabilities can be calculated after integration over appropriate regions in xy-plane.

2. The marginal distributions are

$$f_X(x) = x + 0.5,\quad 0 < x < 1, \qquad = 0, \quad \text{elsewhere,} \qquad \text{and} \qquad f_Y(y) = y + 0.5,\quad 0 < y < 1, \qquad = 0, \quad \text{elsewhere.}$$

So E(X) = E(Y) = 7/12, $E(X^2) = E(Y^2) = 5/12$, V(X) = V(Y) = 11/144, E(XY) = 1/3, Cov(X, Y) = -1/144. Hence V(X + Y) = 5/36, Corr(X, Y) = -1/11.

$$f_{X|Y=y}(x \mid y) = \frac{2(x+y)}{1+2y},\quad 0 < x < 1, 0 < y < 1,$$
$$= 0, \qquad \text{elsewhere.}$$

$$E(X \mid Y = y) = \frac{2(2+3y)}{(1+2y)}, E(X^2 \mid Y = y) = \frac{2(3+4y)}{(1+2y)}, V(X \mid Y = y) = \frac{2(8y^2+4y-1)}{(1+2y)^2}.$$

3. $f_X(x) = \frac{3}{2} - x, 0 < x < 1$ and $f_Y(y) = \frac{3}{2} - y, 0 < y < 1$.

$$f_{X|y}(x \mid y) = \frac{2(2-x-y)}{3-2y}, 0 < x < 1, 0 < y < 1.$$

$$f_{Y|x}(y \mid x) = \frac{2(2-x-y)}{3-2x}, 0 < y < 1, 0 < x < 1.$$

Expectations etc. can be easily calculated now.

4. Use marginal and conditional distributions which are univariate normal.

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6. $f_Y(y) = e^{-y}, y > 0,\quad f_{X|Y=y}(x \mid y) = \frac{1}{y}e^{-x/y}, x > 0, y > 0.$ So E(Y) = 1, V(Y) = 1,

E(X) = EE(X|Y) = E(Y) =1,
V(X) = EV(X|Y) + VE(X|Y) = $E(Y^2)$ + V(Y) = 2 + 1 = 3.,
E(XY) = E{YE(X|Y)} = $E(Y^2)$ = 2, Cov(X, Y) =1.

So , Corr $(X,Y) = \frac{1}{\sqrt{3}}$.

7. Let X be the marks in the first exam and Y in the second exam. Then we have to find the conditional distribution of Y | X = 60 which is univariate normal. Then we can easily find P(Y > 80|X = 60).
When ρ = -0.9, then the conditional distribution of Y | X = 60 will change and the probability will be affected.

8. Similar to Q. 8.

9. $f_X(x) = 1,\ 0 < x < 1,$ and $f_Y(y) = 1,\ 0 < y < 1.$ So E(X) = E(Y) = ½,

$V(X) = V(Y) = 1/12$, Cov $(X, Y) = -\alpha/36$, $\rho_{X,Y} = -\alpha/3$. $-1 \le \alpha \le 1$.
Clearly $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all x, y iff $\alpha = 0$, that is, $\rho_{X,Y} = 0$.

10. (a) Find $P(X \le 2, Y \ge 1)$.
(b) $p_X(0) = 0.210, p_X(1) = 0.298, p_X(2) = 0.277, p_X(3) = 0.215$.
$p_Y(1) = 0.267, p_Y(2) = 0.397, p_Y(3) = 0.302, p_Y(1) = 0.034$.
$E(X) = 1.497$, $V(X) = 1.1$, $E(Y) = 2.103$, $V(Y) = 0.6944$,
$Cov(X, Y) = 0.130809$, $\rho_{X,Y} = 0.1497$.
(iii) 0.6879.
11. The calculations are standard.
12. $P(2X < Y) = ½$, $P(|X + Y| < 1) = 13/16$. Also $X, Y \sim U(-1, 1)$.
So $E(X) = E(Y) = 0$, $V(X) = V(Y) = 1/3$, Cov $(X, Y) = 1/9$, $\rho = 1/3$.