Linear Systems

Notations:

$\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

$$\mathbb{F}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : \begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix} \in \underbrace{\mathbb{R} \text{ or } \mathbb{C}}_{\mathbb{F}} \right\}$$

$$0_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k \times}$$

$$\mathbb{F}^{m \times n} = \left\{ A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} : a_{ij} \in \mathbb{F} \right\}$$

$0_{m \times n}$ = $m \times n$ zero matrix

$$\mathbb{F}^{n^2} = \left\{ A = (a_{ij})_{1\le i,j\le n} : a_{ij} \in \mathbb{F} \right\}$$

I_n = The $n \times n$ identity matrix

O_n = The $n \times n$ zero matrix

<u>Linear System</u>

Given $A \in \mathbb{F}^{m\times n}$

To find $x \in \mathbb{F}^n$ s.t

$$Ax = b$$

for any given $b \in \mathbb{F}^m$

In Particular when

$$b = \theta_m$$

we get

$$Ax = \theta_m$$

Homogeneous System (HS) corresponding to the matrix A

When $b \neq \theta_m$ then the system

$$Ax = b$$

is called a NONHOMOGENEOUS SYSTEM

Simple Properties of HS

$$Ax = \theta_m$$

(1) $x = \theta_n$ is always a sol of the HS

$\because A(\theta_n) = \theta_m$

Homog-Syst always has a sol $x = \theta_n$

This is called THE TRIVIAL SOL

2) Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_{2\times 2}$

HS: $x_1 + x_2 = 0$

$x_1 - x_2 = 0$

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}\right\}$$ gives $x = \theta_2$ the TRIVIAL SOL

This is the ONLY SOL for the HS

EXAMPLE $\quad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

HS: $\quad x_1 - x_3 = 0$
$\quad\quad x_2 - x_3 = 0$

Again $\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right\}$ i.e. $x = \theta_3$ is the TRIVIAL SOL

But observe

$x_1 = 1, \; x_2 = 1, \; x_3 = 1$

$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which is also a sol of the HS

Nonzero sol of the HS

Nontrivial solutions

Homog Syst

May have only TRIVIAL SOL

or May have nontrivial sol

Suppose $Ax = \theta_m$ has a nontrivial sol say $x_H \neq \theta_n$

For any $\alpha \in \mathbb{F}$ if we set

$$u = \alpha x_H$$

Then

$$Au = A(\alpha x_H)$$
$$= \alpha A x_H$$
$$= \alpha \theta_m$$
$$= \theta_m$$

$\Rightarrow$ $u = \alpha x_H$ is a sol of HS $\forall \alpha \in \mathbb{F}$

$\Rightarrow$ (Since $x_H \neq \theta_m$) There are inf no of sol for HS obtained by varying α over $\mathbb{F}$

CONCLUSION The HS $Ax = \theta_m$

EITHER has only TRIVIAL SOL

OR Has inf no. of solutions

Nonhomogeneous System (NHS)

$A \in \mathbb{F}^{m \times n}$ $\quad b \in \mathbb{F}^m$

$$Ax = b$$

SUPPOSE THE SYSTEM HAS A SOL

(i.e the system is CONSISTENT)

How Many?

Suppose NHS has two different sol i.e.

Sol to NHS is not unique

$\Rightarrow \exists\, u, v$ s.t. $Au = b$ and $Av = b$ and $u \neq v$

$$\Rightarrow A(u-v) = Au - Av = b - b = \theta_m$$

$\Rightarrow w = u - v$ satisfies $Aw = \theta_m$

$\Rightarrow w$ is a sol of HS & $w \neq \theta_n$

$\Rightarrow w$ is a NONTRIVIAL sol for HS

CONCLUSION I

SOL to NHS NOT UNIQUE
$\Rightarrow$ HS has NONTRIVIAL Sol

Conversely.

START WITH
Consisten system
$Ax = b$

Suppose
HS has Nontrivial Sol

TST NHS sol is not unique

Proof
Since the system $Ax = b$ IS consistent
$\exists\, u$ s.t $Au = b$ - - (1)

Let $x_H \neq \theta_n$ be a nontrivial
sol for the HS

Let $v = u + x_H$

Clearly $Av = Au + Ax_H$

$= b + \theta_m$

$\Rightarrow Av = b$

$\Rightarrow$ v is a sol of the NHS

Further $v - u = x_H$

$\neq \theta_n$

$\Rightarrow v \neq u$

$\Rightarrow u, v$ are two different sol of the NHS

CONCLUSION 2

HS has Nontrivial sol

$\Rightarrow$ NHS the sol is not unique

Combining the two conclusions we get

<u>RESULT</u> : $A \in \mathbb{F}^{m \times n}$ $b \in \mathbb{F}^m$
$Ax = b$ CONSISTENT

NHS sol is nonunique
$\Longleftrightarrow$ HS has nontrivial sol

Equivalently

NHS has Unique Sol
$\Longleftrightarrow$ HS has only Trivial Sol

So the HS plays a decisive role in determining Uniqueness of the sol of NHS

Look at the NHS

$$Ax = b$$ — Assume consistent

Suppose we know one sol of the NHS, say x_p. This means

$$Ax_p = b \quad \ldots (1)$$

Suppose u is any other sol for the NHS i.e.

$$Au = b$$

and $\quad u \neq x_p$

$v = u - x_p$

We have

$$A v = A(u - x_p)$$
$$= Au - Ax_p$$
$$= b - b$$
$$= 0_m$$

$\Rightarrow$ v is a sol of the homog. syst

$\Rightarrow$ $u = x_p$ + a sol of the h.s

CONCLUSION 1.

x_p a solution of the NHS

$\Rightarrow$ Any other sol of the NHS must be of the form

x_p + a sol of the HS

<u>Conversely</u> Consider any u which is of the form

$$u = x_p + x_H$$ (is a sol of HS)

$$\Rightarrow Au = Ax_p + Ax_H$$

$$= b + \theta_m$$

$$= b$$

$\Rightarrow$ u is a sol of the NHS

<u>CONCLUSION 2</u>

x_p a sol of NHS, x_H a sol of HS

$\Rightarrow$ $u = x_p + x_H$ must be a sol of NHS

RESULT $Ax = b$ consistent

x_p a sol of the NHS

$\Longrightarrow$ Every sol^n NHS is obtained in the form

$x_p + x_H$

by varying x_H over all sol of HS

Finding Solutions of NHS involves Two Parts:

FIRST PART

Finding all solutions of the HS

SECOND PART

Finding a sol for the NHS: x_p

(Then can generate all sol of the NHS by adding to x_p the various sol of the HS)

First we develop strategy for finding all sol of the HS

HOMOGENEOUS SYSTEMS

$A \in \mathbb{F}^{m \times n}$

HS: $Ax = \theta_m$

Problem Find all solutions

Recall This may have only TRIVIAL SOL

OR

This May have inf. no of solution

MAIN TOOL

Elementary Row Operations

(EROs)

The EROs are some operations performed on the rows of a matrix

Three Types of ERO

ERO of Type 1
ERO of Type 2
ERO of Type 3

ERO of Type 1 Row Interchange (Row Exchange)

ERO of Type 2 To the i^{th} row add a multiple of the j^{th} row

ERO of Type 3 The entries of the i^{th} row are all multiplied by a NONZERO $\alpha \in \mathbb{F}$

The idea

Use ERO's on the given matrix A repeatedly to get a new matrix B

Look at $Ax = \theta_m$ HS corr. to A

$Bx = \theta_m$ HS corr to B

It turns out that the HS for A has the same set of sol as the HS for B

Therefore instead of solving the HS $Ax = \theta_m$ we can solve the HS $Bx = \theta_m$

What is the advantage?

If we are clever then we can choose our EROs s.t the matrix B is simple enough to make solving the system $Bx = 0_m$ easy.

For this to be achieved we must

1) Know what simple system B can be?
2) How can we use ERO's effectively to reduce a given A to a simple B?