$$A \in \mathbb{R}^{m \times n}$$

Defined

$$L = A^T A \in \mathbb{R}^{n \times n}$$

$$M = A A^T \in \mathbb{R}^{m \times m}$$

Both are real symm PSD matrices

We found that the decomposition of $\mathbb{R}^n$ & $\mathbb{R}^m$ are as follows.

$\begin{bmatrix} v_1 \\ \vdots \\ v_\rho \end{bmatrix}$ $R_{A^T} = R_L$

$\begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{\nu_A} \end{bmatrix}$ $N_A = N_L$

$\begin{bmatrix} u_1 \\ \vdots \\ u_\rho \end{bmatrix}$ $\underline{R_A = R_M}$

$\begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{\nu_{A^T}} \end{bmatrix}$ $N_{A^T} = N_M$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, shift left, "A"] & \mathbb{R}^m \arrow[l, shift left, "A^T"] \end{tikzcd}$$

<u>Ranks</u>: $\rho_A = \rho_{A^T} = \rho_L = \rho_M$ — We denote rank by ρ

<u>Nullities</u>: $\nu_A = \nu_L$

$\nu_{A^T} = \nu_M$

Bases:

1) $\varphi_1, \varphi_2, \ldots, \varphi_{\nu_A}$ o.n.b. for $\mathcal{N}_A$
   - These are o.n. eigenvectors corresp to the eigenvalue 0 of L

2) $\psi_1, \psi_2, \ldots, \psi_{\nu_{A^T}}$ o.n.b for $\mathcal{N}_{A^T}$
   - These are o.n. eigenvectors corresp to the eigenvalue 0 of M

3) $v_1, v_2, \ldots, v_p$: onb for $\mathcal{R}_{A^T}$
   - These are the o.n. eigenvectors corresp to the positive eigenvalues of L

arranged as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$$

4) $u_1, u_2, \ldots, u_p$ onb for R_A

where $u_j = \frac{1}{s_j} A v_j$, $1 \leq j \leq p$

where
$s_j = \sqrt{\lambda_j}$: (Singular Values of A)

5) $\left\{ \begin{array}{l} A v_j = s_j u_j \\ A^T u_j = s_j v_j \end{array} \right\}$ $1 \leq j \leq p$

How do we use these facts to analyse the various questions about A

and the structure of A

(I) Recall if $A \in$ Real symmetric $A \in R^{n \times n}$

then $\exists$ real orthogonal matrix $U \in R^{n \times n}$

s.t

$$A = UDU^T \quad , \text{ where } D \text{ is diagonal } \in R^{n \times n}$$

Now suppose $A \in R^{m \times n}$

We pick the o.n-bases as above

We construct the matrix

$$U = [u_1 \; u_2 \; \cdots \; u_p \; \psi_1 \; \psi_2 \; \cdots \; \psi_{\nu_{A^T}}] \in R^{m \times m}$$

This is an orthogonal matrix

because columns are o.n-vectors

$$U^T U = I_{m\times m} = U U^T$$

$$U^T = U^{-1}, \quad (U^T)^{-1} = U$$

Similarly if we define

$$V = [v_1 \; v_2 \; \cdots \; v_p \; \varphi_1 \; \cdots \; \varphi_{v_A}] \in \mathbb{R}^{n\times n}$$

This is also orthogonal since columns are o.n. vectors

$\therefore$

$$V^T V = I_{n\times n} = V V^T$$

$$(V^T)^{-1} = V \qquad V^T = V^{-1}$$

We have

$$AV = A\,[v_1 \cdots v_p\ \varphi_1 \cdots \varphi_{\nu_A}]$$

$$= [Av_1\ Av_2 \cdots Av_p\ A\varphi_1 \cdots A\varphi_{\nu_A}]$$

$$= [s_1u_1\ s_2u_2 \cdots s_pu_p\ \theta_m \cdots \theta_m]$$

$$U^TAV = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_p^T \\ \psi_1^T \\ \vdots \\ \psi_{\nu_{A^T}}^T \end{bmatrix} [s_1u_1\ s_2u_2 \cdots s_pu_p\ \theta_m \cdots \theta_m]$$

$$= \left(\begin{array}{cccc|ccc} s_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 & 0 & \cdots & 0 \\ - & - & & - & & - & \\ 0 & - & - & s_p & 0 & \cdots & 0 \\ \hline \end{array}\right.$$

$$\begin{pmatrix} 0 & 0 & - & \Big| & - & & 0 \\ 0 & 0 & - & \Big| & - & & 0 \\ \cdot & - & & \Big| & & & \\ 0 & - & & \Big| & & & 0 \end{pmatrix}$$

$$= \left[\begin{array}{c|c} S_{p\times p} & 0_{p\times(n-p)} \\ \hline 0_{(m-p)\times p} & 0_{(m-p)\times(n-p)} \end{array}\right] = S_A$$

(essentially a diagonal matrix)

where $$S_{p\times p} = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_p \end{pmatrix}$$

$$\Rightarrow \quad U^T A V = S_\lambda$$

$$\Rightarrow \quad \boxed{A = \underset{m\times m}{U}\ \underset{m\times n}{S_A}\ \underset{n\times n}{V^T}}$$

Thus we have a factorization of A into three factors – the two extreme factors are orthogonal matrices & the middle one is "essentially" a diagonal matrix

This is called the SINGULAR VALUE DECOMPOSITION of A (SVD of A)

In case we have $m = n$

We get the SVD as

$$A = \underset{n\times n}{U}\ \underset{n\times n}{S_A}\ \underset{n\times n}{V^T}$$

$$S_A = \left(\begin{array}{c|c} \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_\rho \end{matrix} & 0_{\rho\times n-\rho} \\ \hline 0_{n-\rho\times\rho} & 0_{n-\rho\times n-\rho} \end{array}\right)_{n\times n}$$

$$= \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & 0 & \\ & & s_\rho & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Thus we have a general diagonalization theorem for any $n \times n$ real matrix:

Given $A \in \mathbb{R}^{n\times n}$ $\exists$ orthogonal matrices $U, V \in \mathbb{R}^{n\times n}$ s.t

$$A = U D V^T$$

where D is a diagonal matrix in $\mathbb{R}^{n\times n}$

where $D = \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & 0 & \\ & & s_\ell & & & \\ & 0 & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$

$s_1, s_2, \ldots, s_\ell$ Singular values of A
(ℓ rank of A)

This is a generalization of

(1) the factorization we had for Hermitian matrices and

(2) the diagonalization we had for matrices which were 'diagonalizable' over $\mathbb{C}^n$

(II) $A \in \mathbb{R}^{m \times n}$

We have the bases above

We have

$v_1, v_2, \ldots, v_\ell, \varphi_1, \ldots, \varphi_{\nu_A}$ is an onb for $\mathbb{R}^n$

Therefore

$x \in \mathbb{R}^n \Longrightarrow$ x can be expanded in terms of this o.n.b.

$$\Longrightarrow x = \sum_{j=1}^{\rho} (x, v_j) v_j + \sum_{r=1}^{\nu_A} (x, \varphi_r) \varphi_r$$

$$\Longrightarrow Ax = \sum_{j=1}^{\rho} (x, v_j) A v_j + \sum_{r=1}^{\nu_A} (x, \varphi_r) A\varphi_r$$

$$\Longrightarrow Ax = \sum_{j=1}^{\rho} (x, v_j) s_j u_j \qquad \left(\because A\varphi_r = \theta_m \ \ 1 \le r \le \nu_A \text{ and } Av_j = s_j u_j \right.$$

Thus

$$x \in \mathbb{R}^n \Rightarrow Ax = \sum_{j=1}^{\rho} (x, v_j) s_j u_j \quad \ldots (1)$$

$$(x, v_j) = v_j^T x$$

$$(x, v_j)\, s_j\, u_j = \underline{s_j\, v_j^T x\, u_j}$$

For any $u \in R^m$ & $v \in R^n$ define

$$\underset{}{v \otimes u} = \underset{m\times 1}{u}\ \underset{1\times n}{v^T} \quad (m \times n \text{ matrix in } R^{m\times n})$$

for $x \in \mathbb{R}^n$

$$\longrightarrow \underset{m\times n}{(v \otimes u)}\ \underset{n\times 1}{x} = (u v^T) x$$

$$= u\,(v^T x)$$

$$= (v^T x)\, u \longleftarrow$$

$$\therefore \quad (v_j \otimes u_j)\, x = (v_j^T x)\, u_j$$

$$\therefore \quad s_j\,(v_j^T x)\, u_j = s_j\,(v_j \otimes u_j)\, x$$

Hence (1) can be written as

$$x \in \mathbb{R}^n \Rightarrow Ax = \sum_{j=1}^{p} s_j (v_j \otimes u_j) x$$

$$\Rightarrow Ax = \underbrace{\left\{ \sum_{j=1}^{p} s_j (v_j \otimes u_j) \right\}}_{K \in \mathbb{R}^{m \times n}} x$$

$$x \in \mathbb{R}^n \Rightarrow Ax = Kx$$

$$\Rightarrow A = K$$

$$\Rightarrow \boxed{A = \sum_{j=1}^{p} s_j (v_j \otimes u_j)}$$

SVD (Sum Version)

Each term in the sum is an $m \times n$

matrix of Rank one.

Thus we have

Any $A \in \mathbb{R}^{m \times n}$ of rank ρ can be expressed as the sum of ρ one ranked $m \times n$ matrices in $\mathbb{R}^{m \times n}$

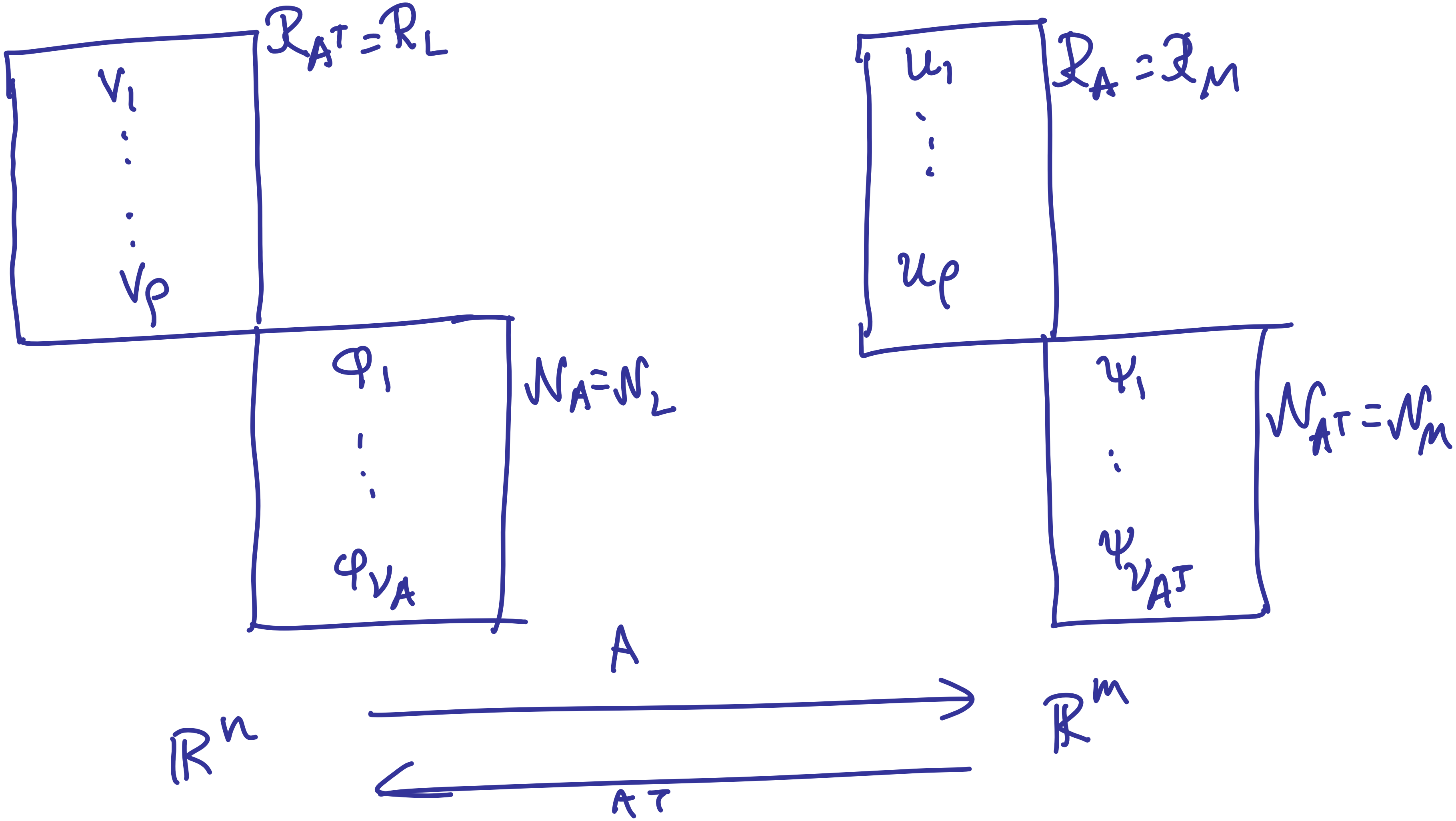
<u>Summary</u>

$A \in \mathbb{R}^{m \times n}$

$L = A^T A \in \mathbb{R}^{n \times n}$ symmetric PSD

$M = A A^T \in \mathbb{R}^{m \times m}$ symmetric PSD

These yield us the following bases (o.n.)



$$u_j = \frac{1}{s_j} A v_j \quad , \quad 1 \le j \le \rho$$

$$s_j = \sqrt{\lambda_j}$$ (the λ_j being the j^{th} pos eigenvalue of L)

ii) (Singular Values of A)

$$A v_j = s_j u_j$$

$$A^T u_j = s_j v_j$$

$V^T = V^{-1}$, $V \ne (V^T)^{-1}$

$\times$ $V^T V = I_{n\times n}$

We defined

$$V = [v_1 \; \cdots \; v_p \; \varphi_1 \; \cdots \; \varphi_{\nu_A}] \in \mathbb{R}^{n\times n}$$ orthogonal

$$U = [u_1 \; \cdots \; u_p \; \psi_1 \; \cdots \; \psi_{\nu_{A^T}}] \in \mathbb{R}^{m\times m}$$ orthogonal

$U^T U = I_{m\times m}$

$U^T = U^{-1}$

Product Decomposition of A (SVD)

$$A = U S_A V^T$$ where

$$S_A = \left[\begin{array}{c|c} S_{p \times p} & 0_{p \times n-p} \\ \hline 0_{m-p \times p} & 0_{m-p \times n-p} \end{array}\right]$$

$$S_{p \times p} = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_p \end{pmatrix}$$

SUM DECOMPOSITION

$$A = \sum_{j=1}^{p} s_j \left(v_j \otimes u_j\right)$$

where $v_j \otimes u_j = u_j v_j^T$

How do we use these bases to analyse the system of equation

$$Ax = b$$

Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

To find $x \in \mathbb{R}^n$ s.t. $Ax = b$

Questions:

1) What is the Consistency Condn?
2) When b satisfies these condns;
   How many sol

If unique find THE SOL

If infinite find all & find unique representative

3) When b does not satisfy these condn -

Least Sq - Sol

Unique

Find The Unique Sol

Inf

Find all & find the unique representat