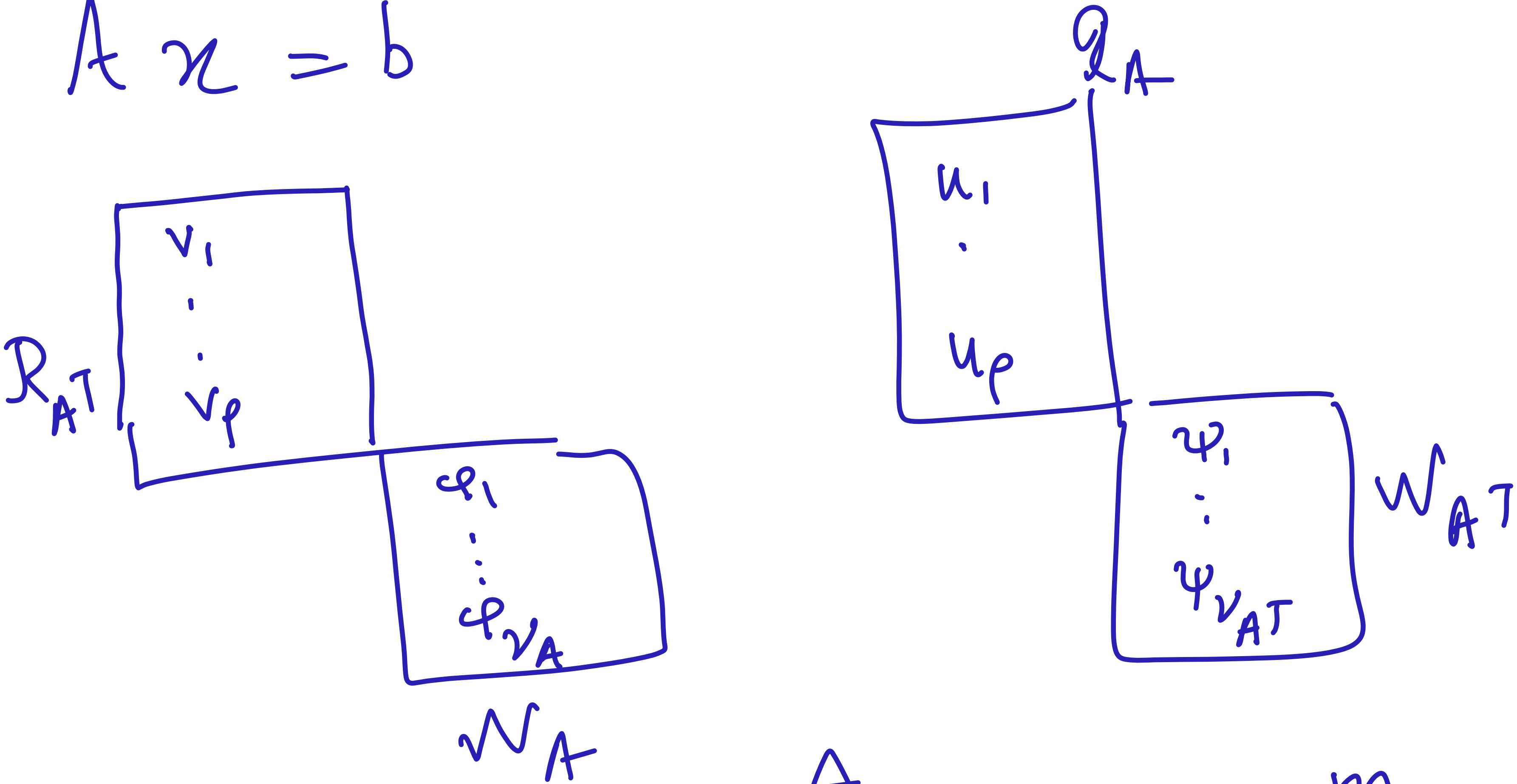
$$Ax = b$$



$$\mathbb{R}^n \underset{A^T}{\overset{A}{\rightleftarrows}} \mathbb{R}^m$$

$$A v_j = s_j u_j \quad , \quad A^T u_j = s_j v_j \quad , \quad 1 \le j \le \rho$$

The Consistency Condn:

$$(b, \psi_j) = 0 \quad , 1 \le j \le \nu_{A^T}$$

($m-\rho$ conditions)

When b satisfies then conditions

Solution to the system

$Ax = b$ exists

(1) solution is unique if $\rho = n$

Unique sol is given by

$$\boxed{x = \sum_{j=1}^{\rho(=n)} \frac{1}{s_j} (b, u_j)\, v_j}$$

(2) If $\rho < n$ then there inf no. of sol. given by

$$\Rightarrow \quad x = \sum_{j=1}^{\rho(<n)} \frac{1}{s_j} (b, u_j)\, v_j + \sum_{k=1}^{\nu_A} \alpha_k \varphi_k$$

where $\alpha_1, \alpha_2, \dots, \alpha_{\nu_A}$ can be chosen arbitrarily in $\mathbb{R}$

Among these inf-sol.

$$\longrightarrow x_{opt} = \sum_{j=1}^{p} \frac{1}{s_j} (b, u_j) v_j$$

(it has the lowest length among all solutions)

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b does not satisfy the consistency condition:

For any $x \in \mathbb{R}^n$, Ax cannot be equal to b

We consider the error

$$e_b(x) = \|Ax - b\|^2$$

Look for $x_l \in \mathbb{R}^n$ s.t

$$e_b(x_l) \leq e_b(x) \quad \forall x \in \mathbb{R}^n$$

Hence $Ax_l \in R_A$, & Ax_l is closest to b in R_A.

Such an x_l is called a least square sol.

But b can be written as

$$b = b_r + b_n$$

where

$$b_r = \sum_{j=1}^{\rho} (b, u_j) u_j$$

$$b_n = \sum_{j=1}^{\nu_{A^T}} (b, \psi_j) \psi_j$$

We know that the vector in R_A closest to b is the orthogonal proj. of b onto R_A — which is b_r

So the least square sol x_l we are looking for must be s.t

$$Ax_l = b_r$$

$\Rightarrow$ Least sq. solutions are solutions of the system

$$Ax = b_r$$

Since $b_r \in R_A$, it satisfies the consistency condition

$$(b_r, \psi_j) = 0 \qquad 1 \leq j \leq \nu_{A^T}$$

Therefore the system

$$Ax = b_R$$

has a sol. Hence we get always a least sq sol when $b \in R_A$ i.e. when b does not satisfy consistency condition.

From our knowledge for the case of the system when rhs satisfies consistency condition, we get

1) when $\rho = n$

unique sol for $Ax = b_R$

$\therefore$ unique least square sol

given by

$$x_\ell = \sum_{j=1}^{\rho} \frac{1}{s_j} (b_R, u_j) v_j$$

However it is easy to see

$$(b_R, u_j) = (b, u_j), \quad 1 \le j \le \rho$$

Unique least sq sol.

$$\longrightarrow \quad x_\ell = \sum_{j=1}^{\rho (=n)} \frac{1}{s_j} (b_R, u_j) v_j$$

2) When $\rho < n$

$$Ax = br$$

has inf no. of sol.

$\therefore$ we have inf. no. of least sq. sol

Given by

$$\longrightarrow \quad x_l = \sum_{j=1}^{\rho(<n)} \frac{1}{s_j} \underbrace{(br, u_j)}_{=(b,u_j)} v_j + \sum_{k=1}^{\nu_A} \alpha_k \varphi_k$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be chosen arbitrarily in $\mathbb{R}$

Among them inf. sol

the sol

$$\longrightarrow (x_l)_{opt} = \sum_{j=1}^{p} \frac{1}{s_j} (b, u_j) v_j$$

In all the cases the final sol we are looking for is of the form

$$\sum_{j=1}^{p} \frac{1}{s_j} (b, u_j) v_j$$

When b satisfies Cons. Condi,

$\rho = n$

represents the unique solution for $Ax = b$

$\rho < n$

represents the unique optimal sol for $Ax = b$

When b does not satisfy consistency condn

$\rho = n$

represents the unique least sq. sol

$\rho < n$

represent the unique optimal least sq. sol

Consistency Condn

$$(b, \psi_j) = 0 \quad 1 \le j \le \nu_{A^T}$$

Comes from the on basis for N_{A^T}

When Cons. Condns satisfied by b

$\rho = n$

The unique sol is determined by the expression

$$\sum_{j=1}^{\rho} \frac{1}{s_j} (b, u_j) v_j$$

Completely determined by the o.n.b for R_{A^T}, R_A and the singular values

$\rho < n$

The inf no of sol are given by ν_A

$$\sum_{j=1}^{\rho} \frac{1}{s_j} (b, u_j) v_j + \sum_{k=1} d_k \varphi_k$$

Completely determined by the o.n.b for R_{A^T}, R_A, and N_A, and singular values

The unique opt. sol. given by

$$\sum_{j=1}^{\ell} \frac{1}{s_j}(b, u_j)\, v_j$$

is completely determined by the onb for R_{A^T}, R_A & the sing. vals.

When b does not satisfy cons cond.

$\rho = n$

unique least sq sol determined by onb for R_{A^T} & R_A and singular values

$\rho < n$

inf no of least sq. sol determined by onb for R_{A^T}, R_A, N_A and singular values

$\downarrow$

Unique optimal least sq. sol is given by the onb for R_{A^T}, R_A and singular values

Let us look at the expression

$$X_{SOL} = \sum_{j=1}^{\rho} \frac{1}{s_j} (b, u_j) v_j$$

$$X_{sol} = \begin{cases} \text{Unique solution} & \text{when } \rho = n \ \& \ b \text{ satisfies cons cond} \\ \text{unique optimal sol} & \text{when } b \text{ satisfies Consist. condition and } \rho < n \\ \text{Unique least sq. sol} & \text{when } b \text{ does not satisfy cons. cond } \& \ \rho = n \\ \text{unique optimal} & \text{when } b \text{ does not satisfy cons cond} \end{cases}$$

least sq sol & $p < n$

$$X_{sol} = \sum_{j=1}^{p} \frac{1}{s_j} (b, u_j) v_j$$

$$= \sum_{j=1}^{p} \frac{1}{s_j} v_j (b, u_j)$$

$$= \sum_{j=1}^{p} \frac{1}{s_j} v_j u_j^T b$$

$$= \left(\sum_{j=1}^{p} \frac{1}{s_j} v_j u_j^T \right) b$$

For each j, $\quad v_j \in \mathbb{R}^n \qquad v_j$ is $n \times 1$

$u_j \in \mathbb{R}^m \qquad u_j$ is $m \times 1$

u_j^T is $1 \times m$

$\therefore$

$\Rightarrow v_j u_j^T$ is $n \times m$ matrix

$\Rightarrow \frac{1}{s_j} v_j u_j^T$ is $n \times m$ matrix

$\Rightarrow \sum_{j=1}^{\rho} \frac{1}{s_j} v_j u_j^T$ is an $n \times m$ matrix

which we shall denote

by $A^\dagger$

and call this as the

PSEUDO INVERSE OF A

$$A^{\dagger} = \sum_{j=1}^{p} \frac{1}{s_j} v_j u_j^T$$

Hence

$$x_{SOL} = A^{\dagger} b$$

What does this mean?

$$\mathbb{R}^n \quad \xrightarrow[x]{I/P} \quad \boxed{A_{m \times n}} \quad \xrightarrow[Ax]{O/P} \quad \mathbb{R}^m$$

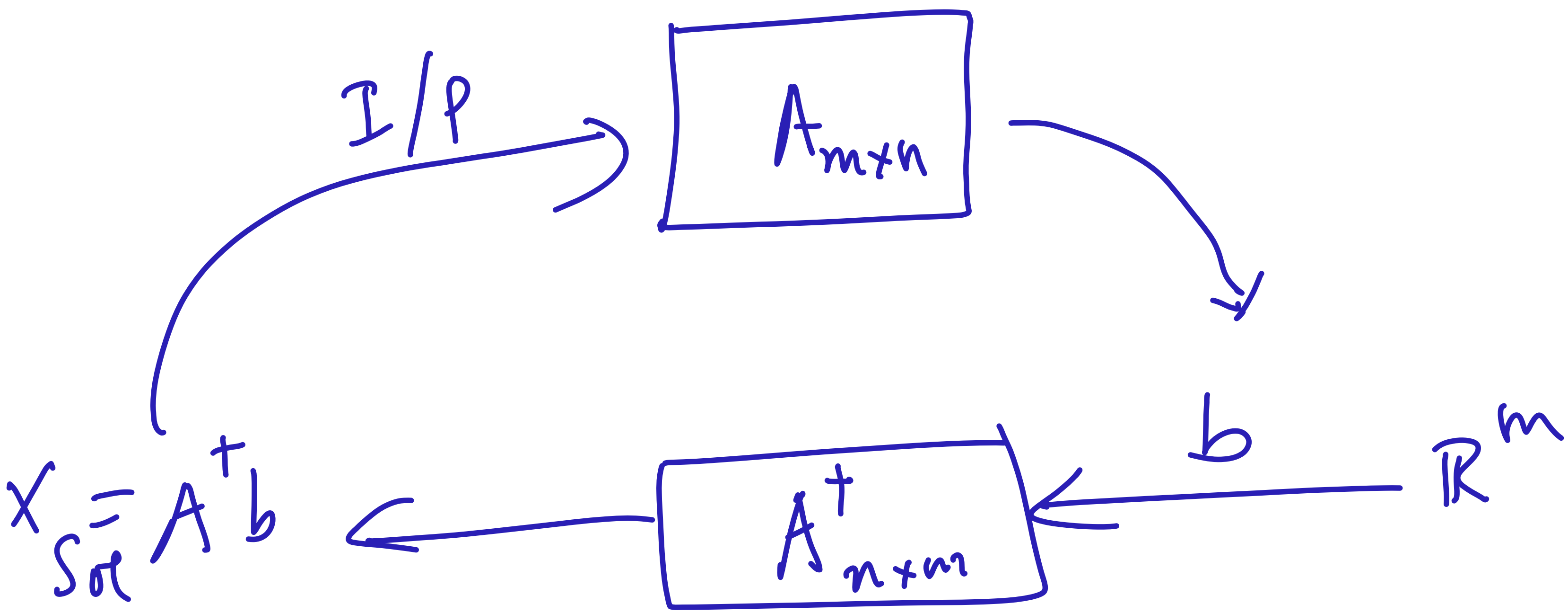
Solving $Ax = b$ means the following:

Given what output b I want

Have to determine the input x

that will give this output

x_{SOL} is "the" required input



$A x_{SOL} = b$ whenever $\exists$ an input giving output b

$A x_{SOL}$ will get closest possible to b whenever there does not exist an input which gives output b exactly

& In both cases x_{SOL} will have the least length among all such reqd. inputs

In the case of a square

matrix $(m = n)$ A

we have the notion of

an inverse A^{-1} of a matrix

(whenever A is invertible).

Now we have also a notion

of Pseudoinverse A^{+}.

If A is invertible square

matrix what is the connection between A^{-1} and $A^{\dagger}$?

We shall show that in this case then $A^{\dagger} = \underline{\underline{A^{-1}}}$

<u>Note therefore</u>

1) A^{-1} makes sense only when A is a square invertible matrix.

But

2) $A^{\dagger}$ makes sense for square matrices whether invertible

or not and also for nonsquare matrices

3) Whenever A^{-1} makes sense A^{+} will be $= A^{-1}$

Thus A^{+} is a generalized notion of inverse of a matrix.

A square matrix $n \times n$ $(m=n)$

A is invertible if Rank $A = n$

$\therefore \quad \underline{\underline{\rho = n}}$

<u>Therefore</u>

$$\underline{\underline{\rho = n = m}}$$

When $m = \rho$ — No consistency condns to be satisfied

$\therefore$ Sol for all $b \in \mathbb{R}^n$ exists to the system $Ax = b$

Sol is unique since $\rho = n$

Unique sol given by

$$x = A^{\dagger} b \quad \forall\, b \in \mathbb{R}^n$$

But $Ax = b$ has unique sol

$x = A^{-1} b \quad \forall b \in \mathbb{R}^n$ $\because$ A is invertible

$$\Rightarrow \quad A^{+} b = A^{-1} b \quad \forall \ b \in \mathbb{R}^n$$

$$\Longrightarrow \quad A^{+} = A^{-1}$$