A $m \times n$ real matrix

$b \in \mathbb{R}^m$

System $Ax = b$

Given b Want to find $x \in \mathbb{R}^n$

s.t $Ax = b$

Question: What are the conditions — if any — that b has to satisfy for the system $Ax = b$ to have a solution?

(Any such condition will be called CONSISTENCY CONDITION(S))

What is/are the consistency condition/s?

Answer:

(1) Rank A = Rank A_{aug}
(by using Elementary Row operations)

OR equivalently

(2) $b \in R_A$

OR equivalently

(3) $b \perp$ all vectors in N_{A^T}

OR equivalently

(4) $b \perp$ all vectors in a basis for N_{A^T}

OR equivalently

(5) $(b, \psi_j) = 0$ where

$\psi_1, \psi_2, \ldots, \psi_{\nu_{A^T}}$ is an o.n.b for N_{A^T}

Having found the consistency conditions, given $b \in \mathbb{R}^m$ we shall ask

Does b satisfy the Cons-condns?

- **YES**
  - Solution to system $Ax = b$ exists
  - ↓ How Many
    - **Unique** → When is it unique? → When $\rho = n$ then sol is unique
    - **Infinitely Many** → When is it inf. many? → When $\rho < n$ then inf no. of sol
- **NO**
  - There is No solution
  - ↓ What can we do?
  - ↓ We can find least square sol.
  - ↓ How many?
    - **Unique** → When is it unique?
    - **Infinitely Many** → When do we get infinitely many sol.?

What is THE solution?

The sol is given by

$$x = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j$$

What are all the solution?

Given by

$$x = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j + \sum_{k=1}^{n} \alpha_k \varphi_k$$

($\alpha_1, \ldots, \alpha_k$ arbitrary)

Unique rep. soln?

$$x_{opt} = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j$$

When $p = n$ l.sq. sol is unique

What is this unique least sq. sol.?

$$x_l = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j$$

When $p < n$ we get inf. many l.sq. sol

What are all these l.sq. sol?

$$x = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j + \sum_{k=1}^{n} \alpha_k \varphi_k$$

$\alpha_1, \ldots, \alpha_k$ arb.

Unique rep!

$$(x_l)_{opt} = \sum_{j=1}^{p} \frac{1}{s_j}(b, u_j) v_j$$

Therefore x_{SOL}

$$x_{SOL} = A^{\dagger} b$$

where

$$A^{\dagger} = \sum_{j=1}^{p} \frac{1}{s_j} v_j u_j^T$$

(Pseudo inverse of the Matrix A)

Second Series of Questions

Change of Variables:

$$Ax = b$$

We shall introduce change of variables,

$$\left| \begin{aligned} x &= Py \\ b &= Pc \end{aligned} \right\}$$ where P is $n \times n$ invertible matrix

Then $Ax = b$ becomes

$$APy = Pc$$

$$\Rightarrow \quad P^{-1}APy = c$$

<u>Question</u> Under what conditions can we find an invertible $n \times n$ matrix P (real) such that

$P^{-1}AP$ is a diagonal matrix?

This leads us to "eigen problems"

Answer

(1) The characteristic polynomial of A,

$$C_A(\lambda) = |\lambda I - A|$$

must factor out as

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \cdots (\lambda - \lambda_k)^{a_k}$$

where $\lambda_1, \ldots, \lambda_k$ are real and distinct, and $a_1, a_2, \ldots, a_k$ are integers ≥ 1 and $a_1 + a_2 + \cdots + a_k = n$,

2) If W_j = Null space of $(A - \lambda_j I)$
$(j = 1, 2, \dots, k)$ then

$$\boxed{\dim W_j = a_j}, \quad 1 \le j \le k$$

(g·m = a·m for $j = 1, 2, \dots, k$)

Given any real A ($n \times n$ matrix)
we ask the question

Does A satisfy the above condition?

YES ← → NO

↓

There exists an invertible real P $(n \times n)$ s.t. $P^{-1}AP$ is a diag. matrix

↓

What is such a P?

↓

Let

$u_1^j, u_2^j, \ldots, u_{a_j}^j$

be a basis for W_j

$(j = 1, 2, \ldots, k)$

↓

We cannot get a real $n \times n$ invertible matrix P such that $P^{-1}AP$ is a diagonal matrix

↓

We ask Can we get two real $n \times n$ invertible matrices Q and P s.t. $Q^{-1}AP$ is a diagonal matrix?

↓

YES

↓

What are Q & P

Then we set

$$\rightarrow P = \left(u^1_1, \cdots u^1_{a_1}, u^2_1 \cdots u^2_{a_2} \cdots \right)$$

Then

$$P^{-1} A P = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \lambda_k \end{pmatrix}$$

(with a_1 copies of λ_1, …, a_n copies of λ_k)

$$\downarrow$$

$$Q = U = [u_1 \, u_2 \cdots u_\rho \, \psi_1 \cdots \psi_{\nu_{A^T}}]$$

$$(\& \therefore Q^{-1} = U^T)$$

$$P = V = [v_1 v_2 \cdots v_\rho \varphi_1 \cdots \varphi_{\nu_A}]$$

$$Q^{-1} A P = U^T A V$$

$$= \begin{pmatrix} s_1 & & & \\ & \ddots & & 0 \\ & & s_\rho & \\ 0 & & & 0 \\ & & & & \ddots \end{pmatrix}$$

(SVD)

$$\downarrow$$

This can be generalized to rectangular matrices

$Q = U \quad m \times m$

$P = V \quad n \times n$

$$U^T A V = \begin{bmatrix} S_{p \times p} & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

$$S_{p \times p} = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_p \end{pmatrix}_{p \times p}$$

General SVD

Third Question

$v \in \mathbb{R}^n \quad u \in \mathbb{R}^m \qquad (v, u \text{ nonzero})$

We define order — tensor product —

$$u \otimes v = \underset{n\times 1}{v}\,\underset{1\times m}{u^T} \quad n\times m \text{ mat}$$

$$v \otimes u = \underset{m\times 1}{u}\,\underset{1\times n}{v^T} \quad m\times n \text{ matrix}$$

$v \otimes u$ is a rank 1 matrix

If $v_1, v_2, \ldots, v_p$ are l.i. vectors in R^n

$u_1, u_2, \ldots, u_p$ – – – – R^m

$s_1, s_2, \ldots, s_p$ are positive real numbers

Then

$$\longrightarrow \sum_{j=1}^{p} s_j v_j \otimes u_j = \sum_{j=1}^{p} s_j u_j v_j^T$$

is the sum of p one ranked matrices and has rank p

Question: Can we write every $m \times n$ real matrix of rank p

as the sum of p one ranked matrices:

ANSWER: YES

The SVD (sum form) gives such a decomposition:

$$A = \sum_{j=1}^{p} s_j u_j v_j^T = \sum_{j=1}^{p} s_j v_j \otimes u_j$$

The Main Ingredients for achieving all these goals

the four o.n. bases

$v_1, v_2, \ldots, v_p$ for R_{A^T}

$\varphi_1, \varphi_2, \ldots, \varphi_{\nu_A}$ for N_A

$u_1, u_2, \ldots, u_p$ for R_A

$\psi_1, \psi_2, \ldots, \psi_{\nu_{A^T}}$ for N_{A^T} and

$s_1, s_2, \ldots, s_p$ the singular values.

The Way to get there

$$L = A^T A$$

L is Pos. SEM. DEF

Its eigvalues can be arranged as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0 = \lambda_{p+1} = \cdots = \lambda_n$$

$$\begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_p & \lambda_{p+1} & \lambda_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_p & q_1 & q_k \end{array}$$

$$s_1 = \sqrt{\lambda_1} \quad s_2 = \sqrt{\lambda_2} \quad - \; - \quad s_p = \sqrt{\lambda_p}$$

$$u_j = \frac{1}{s_j} A v_j \qquad A v_j = s_j u_j$$

Get

$$A^T u_j = s_j v_j$$

Get $\psi_1, \ldots, \psi_{\nu_A}$ o.n basis for N_{A^T} by solving

$$A^T x = \theta_m$$

|| Main Computations || $\longrightarrow$ Computation of eigenvalues and eigenvectors of Positive Semi definite matrices

---

<u>What about Complex Matrices?</u>

Analysis is similar to the real case

The changes needed are :

Replace Transpose by Transpose Conjugate

T $\qquad$ $*$

Real	Complex
$(x, y) = y^T x$	$(x, y) = y^* x$
A $m \times n$	$\mathbb{C}$ $m \times n$
R_{A^T}, N_A in $\mathbb{R}^n$ | R_A, N_{A^T} in $\mathbb{R}^m$	R_{A^*}, N_A in $\mathbb{C}^n$ R_A, N_{A^*} in $\mathbb{C}^m$

$\mathcal{R}_{A^T}, \mathcal{N}_A$ orth comp in $\mathbb{R}^n$	$\mathcal{R}_{A^*}, \mathcal{N}_A$ orth comp in $\mathbb{C}^n$
$\mathcal{R}_A, \mathcal{N}_{A^T}$ " " $\mathbb{R}^m$	$\mathcal{R}_A, \mathcal{N}_{A^*}$ " "
$v_1, v_2, \dots, v_\rho$ o.n.b. for $\mathcal{R}_{A^T}$	$\to$ $v_1, \dots, v_\rho$ o.n.b for $\mathcal{R}_{A^*}$
$u_1, u_2, \dots, u_\rho$ " $\mathcal{R}_A$	$u_1, \dots, u_\rho$ onb for $\mathcal{R}_A$
$\varphi_1, \dots, \varphi_{\nu_A}$ " $\mathcal{N}_A$	$\varphi_1, \dots, \varphi_{\nu_A}$ onb for $\mathcal{N}_A$
$\psi_1, \dots, \psi_{\nu_{A^T}}$ " $\mathcal{N}_{A^T}$	$\psi_1, \dots, \psi_{\nu_{A^T}}$ " " $\mathcal{N}_{A^*}$

$L = A^T A$ | $L = A^{\#} A$

Pos. Sem Def

Pos Semd