Homog System

Plays an imp role

1) in deciding the uniqueness of the sol of NHS

& 2) finding all sol of NHS knowing one particular sol of the NHS

MAIN TOOL

Elementary Row operations (ERO)

ERO of Type 1

Row Exchange

Keeps all but two rows unaltered & Interchanges the position of these two rows

If we interchange i^{th} row & j^{th} row positions we shall denote this by R_{ij}

Example

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A_1$$

$A \in \mathbb{F}^{m \times n} \xrightarrow{R_{ij}} A_1$

Some Simple Properties of ERO of Type I

i) ERO of Type I is "invertible"

& $(R_{ij})^{-1} = R_{ij}$ an ERO of Type I

$$\begin{tikzcd} A \arrow[r, "R_{ij}"] & A_1 \arrow[l, bend left, "R_{ij}"] \end{tikzcd}$$

ii) Let us look at the ex

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}_{2 \times 3} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A_1$$

Look at the HS corresponding to A & A_1

$Ax = \theta_2$	$A_1x = \theta_2$
$x_2 + 2x_3 = 0$	$3x_1 + 2x_2 + x_3 = 0$
$3x_1 + 2x_2 + x_3 = 0$	$x_2 + 2x_3 = 0$

So both systems are same

only the order of the equns. is changed

Conclusion: The HS $Ax = \theta_2$ & $A_1x = \theta_2$ have the same set of sols

In general if $A \in \mathbb{F}^{m \times n}$

$$A \xrightarrow{R_{ij}} A_1$$

then the HS $Ax = \theta_m$ and $A_1 x = \theta_m$

have the same set of sols

iii) Ex: $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}_{2\times 3} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A_1$

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E$

Look at

$$EA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A_1$$

$\therefore$ Premultiplication of A by E
produced the same effect
as the ERO

<u>CONCLUSION</u> $\quad A \in \mathbb{F}^{m \times n}$

$$A \xrightarrow{R_{ij}} A_1$$

Then look at $\quad I_m \xrightarrow{R_{ij}} E$

Then $\quad EA = A_1$

<u>ERO of Type 2</u>

Keep all rows except one row unchanged
(Let us say all rows except j^{th} row

are kept unchanged)

To the j^{th} row add α times the i^{th} Row

We write this as

$$A \xrightarrow{R_j + \alpha R_i} A_1 \qquad \alpha \in \mathbb{F}$$

Example

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}_{3\times 3} \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} = A_1$$

Simple Properties of ERO type 2

i) ERO of Type 2 is invertible

$$\left(R_j + \alpha R_i\right)^{-1} = R_j + (-\alpha) R_i$$

<u>Ex</u>

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} = A_1$$

$$A_1 \xrightarrow{R_3 - 2R_1} A$$

The inverse of an ERO of Type 2 is again an ERO of Type 2

ii) Ex:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} = A_1$$

HS corr. to A & A_1

$Ax = \theta_3$

$$\left.\begin{aligned} x_1 + x_2 \quad &= 0 \\ 2x_1 \quad + x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

$A_1 x = \theta_3$

$$\left.\begin{aligned} x_1 + x_2 \quad &= 0 \\ 2x_1 \quad + x_3 &= 0 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

The two HS $Ax = \theta_3$ & $A_1 x = \theta_3$ have the same set of solution

In general

$A \in \mathbb{F}^{m\times n}$

$A \xrightarrow{R_j + \alpha R_i} A_1$

Then the HS $Ax = \theta_m$ & $A_1 x = \theta_m$

have the same set of soln.

iii) Example

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}_{3\times 3} \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} = A_1$$

$$I_3 \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = E$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verify

$$EA = A_1$$

The ERO of Type 2 $R_j + \alpha R_i$ can be effected by EA when

$$I_m \xrightarrow{R_j + \alpha R_i} E$$

SUMMARIZE

Three Types of EROS

Type 1 R_{ij}

Type 2 $R_j + \alpha R_i$

They are invertible. Inverse is again same type ERO

They do not change the set of HS

The effect can be achieved by premultiplying by an "E"

Third Type ERO

All but one row remain unaltered (say except the i^{th} row)

Multiply every entry in the i^{th} row by $\alpha \in \mathbb{F}$ (where $\alpha \neq 0$)

Ex: $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}_{2\times 3} \xrightarrow{(-3)R_2} \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} = A_1$

In general αR_i — (Multiply i^{th} row by α)

i) Look at the systems (H)

$$Ax = \theta_2$$

$$x_1 - x_3 = 0$$
$$2x_1 + x_2 + 2x_3 = 0$$

$$A_1 x = \theta_2$$

$$x_1 - x_3 = 0$$
$$-6x_1 - 3x_2 - 6x_3 = 0$$

Both have same set of sol

In general

$A \in \mathbb{F}^{m \times n}$

$$A \xrightarrow{\alpha R_i} A_1 \quad (\alpha \neq 0)$$

$\Rightarrow$ The HS $Ax = \theta_m$ & $A_1 x = \theta_m$ both have the same set of sol

ii) ERO of Type 3 is invertible

$$(\alpha R_i)^{-1} = \alpha^{-1} R_i$$

This is again an ERO of Type 3

$$A \xrightarrow{\alpha R_i} A_1$$
$$A_1 \xrightarrow{\alpha^{-1} R_i} A$$

iii) Ex:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}_{2\times 3} \xrightarrow{(-3)R_2} \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} = A_1$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{(-3)R_2} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = E$$

$$EA = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix}$$

$$= A_1$$

The ERO of type 3 can be implemented by premultiplying A by "an E"

All ERO's have

1) They are invertible
Inverse is of the same type.

2) They do not alter the sol of the HS

3) They can be effected by premultiplication of A by an E

$A \in F^{m \times n}$

$\mathcal{E}$: The set of all EROs on $m \times n$ matrices

I_m : The $m \times m$ identity matrix.

$E_0 \in \mathcal{E}$

$I_m \xrightarrow{E_0} E \rightarrow$ elementary matrices

An $m \times m$ elementary matrix is a matrix obtained by applying an ERO on I_m.

ERO's are effected by premultiplying A by elementary matrices

Look at $\mathbb{F}^{m \times n}$

$A, B \in \mathbb{F}^{m \times n}$

Suppose there are ERO's $E_1, E_2, \ldots, E_k \in \mathcal{E}$

s.t

$$A \xrightarrow{E_1} A_1 \xrightarrow{E_2} A_2 \to \cdots \xrightarrow{E_n} B$$

Then we say

A is ROW EQUIVALENT To B

and

symbolically we write

$$A \overset{R}{\sim} B$$

Observe the following properties

1) $A \overset{R}{\sim} A$ Reflexivity

2) Since each ERO is invertible and again an ERO we see

that

$$A \overset{R}{\sim} B \iff B \overset{R}{\sim} A$$

(Symmetry)

3) $A \overset{R}{\sim} B$ <u>and</u> $B \overset{R}{\sim} C$

$\Rightarrow A \xrightarrow{E_1} \cdots \xrightarrow{E_k} B$

and

$B \xrightarrow{F_1} \cdots \xrightarrow{F_s} C$

$\Rightarrow A \xrightarrow{\text{Finite No of ERO}} C$

$\Rightarrow A \overset{R}{\sim} C$

$$A \overset{R}{\sim} B \ \& \ B \overset{R}{\sim} C \Rightarrow A \overset{R}{\sim} C$$

(TRANSITIVITY)

Hence Row Equivalence is
an EQUIVALENCE RELATION
on $\mathbb{F}^{m \times n}$

Suppose

$$A \overset{R}{\sim} B$$

$$Ax = \theta_m \qquad Bx = \theta_m$$

Since ERO's do not alter the set of sol of the HS we see that the two systems have same set of sol

Conclusion

$A \overset{R}{\sim} B \Rightarrow$ The Homog Systems $Ax = 0_m$ & $Bx = 0_m$ have the same set of sol.

How do we exploit this?

Idea Start from A
Apply suitable ERO's step by step to A to get a Row equivalent matrix B

Then we can solve

$$Bx = 0_m$$

instead of $Ax = 0_m$

<u>Idea</u> Choose ERO' so that easy to solve $Bx = 0_m$?