HS

$A \in \mathbb{F}^{m \times n}$

$Ax = \theta_m$

$A \xrightarrow{ERO} A_R$

$A_R x = \theta_m$ — Solve

(Pivotal Variables are eliminated in terms of NPV) — (NPV are chosen arbitrarily

<u>NHS</u> $b \in \mathbb{F}^m$

$$Ax = b$$

$$A \xrightarrow{EROs} A_R$$

$Ax = b$ equivalent to $A_R x = b$?

<u>Example</u> $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{2\times 3}$

$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$A \xrightarrow{ERO} A_1$

$A \xrightarrow{R_{12}} A_1$

$A_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

<u>System $Ax = b$</u>

$$2x_1 + 3x_2 - x_3 = 1$$
$$x_1 - x_2 + x_3 = 2$$

<u>System $A_1 x = b$</u>

$$x_1 - x_2 + x_3 = 1$$
$$2x_1 + 3x_2 - x_3 = 2$$

Not equivalent

Observe $x_1 = \frac{7}{5}$, $x_2 = -\frac{3}{5}$, $x_3 = 0$

is a sol of the system $Ax = b$

but not the system $A_1 x = b$

What does ERO do?

An ERO on A

→ same as similar operation

on the lhs of the equations

In HS All rhs are 0

– ERO's keep them as 0

Hence $Ax = \theta_m \sim A_R x = \theta_m$

NHS $Ax = b$

ERO's alter the rhs

We must perform on b whatever ERO's we perform on A

Augmented Matrix

$A \in \mathbb{F}^{m \times n}$ $b \in \mathbb{F}^{m}$

NHS: $Ax = b$

$$A_{aug} \overset{\text{def}}{=} (A|b) \in \mathbb{F}^{m \times (n+1)}$$

Ex: $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{2\times 3}$ $\quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$Ax = b$

$$A_{aug} = \left(\begin{array}{ccc|c} 2 & 3 & -1 & b_1 \\ 1 & -1 & 1 & b_2 \end{array}\right)_{2\times 4}$$

General Strategy for NHS

$Ax = b$

$A \xrightarrow{\text{EROs}} A_R$ (RRE form of A)

$b \xrightarrow{\text{same ERO's}} \tilde{b}$

$\overset{\text{NHS}}{Ax = b}$ has same sol as $\overset{\text{NHS}}{A_R x = \tilde{b}}$

$A_{aug} = (A \mid b) \xrightarrow{\text{ERO's}} (A_R \mid \tilde{b})$

Look at $A_R x = \tilde{b}$

$(A_R \mid \tilde{b})$

Suppose Row Rank of A is p

$\implies A_R$ has p nonzero rows
& $m-p$ zero rows

The $\rho+1, \rho+2, \ldots, m$ -- rows of A_R will be zero

$$A_R x = \tilde{b}$$

$(\rho+1)^{th}$ eq : $0x_1 + 0x_2 + \cdots + 0x_n = \tilde{b}_{\rho+1}$

- - -

m^{th} eq : $0x_1 + 0x_2 + \cdots + 0x_n = \tilde{b}_m$

If any of $\tilde{b}_{\rho+1}, \tilde{b}_{\rho+2}, \ldots, \tilde{b}_m$ is not zero say $\tilde{b}_r \neq 0$ $(\rho+1 \leq r \leq m)$

we get the r^{th} eqn as

$$0x_1 + 0x_2 + \cdots + 0x_n = \tilde{b}_r \neq 0$$

$\Rightarrow$ No $x_1, x_2, \ldots, x_n$ can satisfy this equation.

Conclusion

$Ax = b$ is consistent

$\implies \tilde{b}_{p+1} = \cdots = \tilde{b}_m = 0$

$\implies A_{aug} \xrightarrow{ERO} (A_R \mid \tilde{b}) = (A_{aug})_{RREF}$

$\implies$ Row Rank of A_{aug}

= Row Rank of A

$$\boxed{\begin{array}{l} Ax = b \text{ consistent} \\ \implies \text{Row Rank } A_{aug} = \text{Row Rank } A \end{array}}$$

Conversely Suppose

Row Rank A_{aug} = Row Rank A

$(\rho+1)^{th}$ eqn onwards (in $A_R x = \tilde{b}$)

will just be

$$\left.\begin{array}{l} 0 = 0 \\ 0 = 0 \\ \vdots \\ 0 = 0 \end{array}\right\}$$

In the first ρ rows

1st non zero row

$$x_{k_1} + \cdots \cdots = \tilde{b}_1$$

2nd:

$$x_{k_2} + \cdots = \tilde{b}_2$$

$$\tilde{x}_{k_\ell} + \cdots = \tilde{b}_p$$

If we choose

$x_{k_1} = \tilde{b}_1$, $x_{k_2} = \tilde{b}_2$, ..., $x_{k_\ell} = \tilde{b}_p$

& all other $x_j = 0$

$\therefore$ We have

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $x_{k_j} = \tilde{b}_j$ $j = 1, 2, \ldots, p$

$x_j = 0$ if $j \neq \cancel{k}_i$

is a sol of $A_R x = \tilde{b}$

& hence a sol of $Ax = b$

Conclusion

Row Rank A_{aug} = Row Rank A

$\Longrightarrow$ $Ax = b$ is consistent

Theorem: $A \in \mathbb{F}^{m \times n}$ $b \in \mathbb{F}^{m}$

$A_{aug} = (A|b)$

$Ax = b$ is consistent

$\Longleftrightarrow$ Row Rank A = Row Rank A_{aug}

When this criterion is met

we found

$$x_p = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ where

$$x_{k_i} = \tilde{b}_i \qquad 1 \le i \le p$$

$$x_j = 0 \qquad \text{if } j \neq k_i, \ 1 \le i \le p$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_p}$ are the PIVOTAL VARIABLES decided by A_R

The general sol of the NHS

$$Ax = b$$

will then be

$$x = x_H + x_p$$

where x_H is found by

solving $A_R x = \theta_m$

& x_p is as above

CONCLUSION

$A \in \mathbb{F}^{m \times n}$ $\quad b \in \mathbb{F}^m$

To solve NHS $Ax = b$

CRITERION

$A_{aug} = (A \mid b)$

$A \xrightarrow{\text{ERO}} A_R \qquad b \xrightarrow{\text{Same ERO's}} \tilde{b}$

$A_{aug} \xrightarrow{\text{ERO's}} (A_R \mid \tilde{b})$

Row Rank A = Row Rank A_{aug}

If this is satisfied Gen Sol is

$x = x_H + x_p$

To find x_H : Solve $A_R x = 0_m$ (Eliminate PV in terms of NPV)

x_p : As above

NOTE: If Row Rank A_{aug}

$\neq$ Row Rank A

We only know that the system is not consistent i.e there is no sol. We still do not know what to do with such a system.

For a NHS

(1) What is the criterion that b should satisfy for $Ax=b$ to have a sol?

(Row Rank A = Row Rank A_{aug})

Does b satisfy this Criterion

YES

There is a sol

How Many

one

When

($Ax = \theta_m$ has only Trivial Sol)

What is the sol

$x = x_p$

infinite

When

($Ax = \theta_m$ has NONTRIVIAL Sol)

What are the Sol ?

$(x = x_H + x_p)$

NO

No sol Exists

What do we do?

?!

Any criterion to choose a unique rep. ?!

EXAMPLE

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$A_R = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we look at $Ax = b$

We solve $A_R x = \tilde{b}$

Recall

$$A \xrightarrow[\substack{R_3+R_1 \\ R_4-R_1}]{R_2-R_1} \xrightarrow[R_4+2R_2]{R_3-R_2} \xrightarrow{R_4-R_3}$$

$$\xrightarrow{R_1-2R_2} \xrightarrow[R_2+R_3]{R_1-R_3} A_R$$

If we apply on b these operations in the same order

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$b \xrightarrow{\text{EROs}} \begin{pmatrix} b_1 - b_2 - b_3 \\ b_1 + b_3 \\ 2b_1 - b_2 + b_3 \\ -5b_1 + 3b_2 - b_3 + b_4 \end{pmatrix} = \tilde{b}$$

Have to solve

$$A_R x = \tilde{b}$$

Cons. Condition

Rank A = Rank Aaug demands

$$-5b_1 + 3b_2 - b_3 + b_4 = 0$$

i.e. $\boxed{b_4 = 5b_1 - 3b_2 + b_3}$ -- [C]

If b satisfies this condition then
sol to $A_R x = \tilde{b}$ exists and

hence sol to $Ax = b$ exists

Suppose b satisfies [C]

System $Ax = b$ has sol

$$x_P = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad \begin{aligned} x_1 &= \tilde{b}_1 = b_1 - b_2 - b_3 \\ x_3 &= \tilde{b}_2 = b_1 + b_3 \\ x_5 &= \tilde{b}_3 = 2b_1 - b_2 + 2b_3 \\ x_2 &= 0 \\ x_4 &= 0 \end{aligned}$$

(Last lecture)

x_H sol of $A_R x = \theta_m$

$$x_H = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} ; \quad \alpha, \beta \in \mathbb{F}$$

General sol of $Ax=b$ is

$$x = x_H + x_p$$

For example if we had chosen

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

Then we see [C] is satisfied

Sol exists.

Sol $x = x_H + x_p$

$$x_p + x_H = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

If $b = \begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix}$

[C] i) not satisfied
So No sol.