$$Ax = b \qquad A \in \mathbb{F}^{m \times n}$$

$$b \in \mathbb{F}^m$$

Vector Space

NHS

$$Ax = b$$

Two column matrices

$b \in \mathbb{F}^m$ given

& $x \in \mathbb{F}^n$ To be found

k : Positive integer

$$\mathbb{F}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{F} \right\}$$

$$\mathbb{R}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{R} \right\}$$

$x, y \in \mathbb{R}^k$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}, \quad x_j, y_j \in \mathbb{R}$$

$x_1 \in \mathbb{R}$, $y_1 \in \mathbb{R}$ We can get $x_1 + y_1 \in \mathbb{R}$

For each i, $1 \le i \le k$,

$$x_i + y_i \in \mathbb{R}$$

$$\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_k + y_k \end{pmatrix}_{k\times 1} \in \mathbb{R}^k$$

Hence addition $+$ in $\mathbb{R}$ induces an $+$ in $\mathbb{R}^k$ as follows:

$$x + y \stackrel{\text{def}}{=} \begin{pmatrix} x_1 + y_1 \\ \vdots \\ \vdots \\ x_k + y_k \end{pmatrix} \in \mathbb{R}^k$$

<u>Note</u>: LHS $+$ addition in $\mathbb{R}^k$

RHS $+$ " in $\mathbb{R}$

First Major Operation on $\mathbb{R}^k$

$+$ induced by the $+$ on $\mathbb{R}$

↳ Addition on $\mathbb{R}^k$

Properties of Addition $(+)$ on $\mathbb{R}^k$

(1) $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$; $y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$

(0) $x, y \in \mathbb{R}^k \Rightarrow x + y \in \mathbb{R}^k$

$x, y, z \in \mathbb{R}^k$

$$(x+y) + z = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_k + y_k \end{pmatrix} + \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$$

$$= \begin{pmatrix} (x_1+y_1)+z_1 \\ (x_2+y_2)+z_2 \\ \vdots \\ (x_k+y_k)+z_k \end{pmatrix}$$

$$= \begin{pmatrix} x_1+(y_1+z_1) \\ \vdots \\ x_k+(y_k+z_k) \end{pmatrix}$$ ($\because$ of Associativity of $+$ in $\mathbb{R}$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} + \begin{pmatrix} y_1+z_1 \\ \vdots \\ y_k+z_k \end{pmatrix}$$ (by def of $+$ in $\mathbb{R}^k$)

))

$$= x+(y+z)$$

$+$ on $\mathbb{R}^k$ is ASSOCIATIVE, i.e.,

$$(x+y)+z = x+(y+z) \quad \forall\, x, y, z \in \mathbb{R}^k$$

2) $\theta_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$

$$x + \theta_k = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \theta_k + x \quad \forall x \in \mathbb{R}^k$$

$$\boxed{\begin{array}{l} \exists\ \theta_k \in \mathbb{R}^k \text{ s.t} \\ \quad x + \theta_k = x = \theta_k + x \\ \quad \forall x \in \mathbb{R}^k \end{array}}$$

3) $x \in \mathbb{R}^k$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

$x_1 \in \mathbb{R} \quad \therefore \quad -x_1 \in \mathbb{R}$

$x_2 \in \mathbb{R} \qquad -x_2 \in \mathbb{R}$

$\vdots$

$x_k \in \mathbb{R} \qquad -x_k \in \mathbb{R}$

Form a new $k \times 1$ matrix

$$(-x) \stackrel{\text{def}}{=} \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix} \in \mathbb{R}^k$$

Clearly,

$$x + (-x) = \theta_k = (-x) + x \quad \forall x \in \mathbb{R}^k$$

We say $\mathbb{R}^k$ forms a Group

With the operation $+$

Further

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ \vdots \\ x_k + y_k \end{pmatrix} = \begin{pmatrix} y_1 + x_1 \\ \vdots \\ y_k + x_k \end{pmatrix}$$

($\because$ of Commutativity of $+$ in $\mathbb{R}$)

$$= y + x$$

$$\boxed{\begin{array}{l} x + y = y + x \quad \forall\, x, y \in \mathbb{R}^k \\ \text{ADDITION IS COMMUTATIVE in } \mathbb{R}^k \end{array}}$$

$(\mathbb{R}^k, +)$ is a Commutative Group.

(Abelian Group)

Generalize

Let V be any Nonempty set

Let $+$ be a rule of combining an $x \in V$ with a $y \in V$ to produce an element in V which we denote by $x+y$ s.t

(0) $x, y \in V \implies x+y \in V$
(V is closed w.r.t $+$)

(1) $x, y, z \in V \implies (x+y)+z = x+(y+z)$
(Associativity of $+$)

(2) $\exists\, \theta_V \in V$ s.t.

$$x + \theta_V = x = \theta_V + x \quad \forall\, x \in V$$

(3) $\forall\, x \in V \quad \exists\, (-x) \in V$ s.t

$$x + (-x) = \theta_V = (-x) + x$$

(4) $x, y \in V \Rightarrow x + y = y + x$

(Commutativity)

Then we say $(V, +)$ is an Abelian group

Second Major Operation in $\mathbb{R}^k$

$x \in \mathbb{R}^k$ $\qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \; x_j \in \mathbb{R}$

Take any $\alpha \in \mathbb{R}$

$\alpha x_1 \in \mathbb{R}$

$\alpha x_2 \in \mathbb{R}$

$\vdots$

$\alpha x_k \in \mathbb{R}$

Form a new element in $\mathbb{R}^k$

$$\alpha x \overset{\text{def}}{=} \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_k \end{pmatrix} \in \mathbb{R}^k \qquad \text{(scalar multiplication)}$$

SCALAR MULT.

a rule which combines an $\alpha \in \mathbb{R}$ with an $x \in \mathbb{R}^k$.

Properties

(0) $\alpha \in \mathbb{R},\ x \in \mathbb{R}^k \implies \alpha x \in \mathbb{R}^k$

(1) $$(\alpha+\beta)x = \begin{pmatrix} (\alpha+\beta)x_1 \\ \vdots \\ (\alpha+\beta)x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta x_1 \\ \vdots \\ \alpha x_k + \beta x_k \end{pmatrix}$$ (DISTRIBUTIVITY mult. in $\mathbb{R}$ over $+$)

$$= \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_k \end{pmatrix} + \begin{pmatrix} \beta x_1 \\ \beta x_2 \\ \vdots \\ \beta x_k \end{pmatrix}$$

① $(\alpha+\beta) x = \alpha x + \beta x \quad \forall \alpha, \beta \in \mathbb{R}$
$\forall x \in \mathbb{R}^k$

(2) $(\alpha\beta) x = \alpha(\beta x)$

(3) $1 x = x \quad \forall x \in \mathbb{R}^k$

$$\boxed{(4)\ \alpha(x+y) = \alpha x + \alpha y \quad \forall \alpha \in \mathbb{R},\ \forall x, y \in \mathbb{R}^k}$$

Generalize all these we get the Notion of a Vector Space

$\mathbb{R}^k \longrightarrow V$ nonempty set

$+ \longrightarrow +$

Rules for $+ \longrightarrow$ Rules $+$

(Scalar Mult Multiply by $\mathbb{R}$) $\longrightarrow$ $\mathbb{F}$ field, Multiplication of a V element with an $\mathbb{F}$ element

Rules for scalar mult $\longrightarrow$ Rules for scalar mult

DEFINITION

Let V be an arbitrary Nonempty set

Let $\mathbb{F}$ be any field

Let $+$ be a rule of combining $x, y \in V$
to get $x+y$

Let $\cdot$ be a rule to combine an $\alpha \in \mathbb{F}$
with an $x \in V$

Such that

(1) $x, y \in V \Longrightarrow x+y \in V$
(V is closed w.r.t $+$)

(2) $x, y, z \in V \Longrightarrow (x+y)+z = x+(y+z)$

($+$ is associative on V)

(3) $\exists\ \theta_V \in V$ s.t

$$x + \theta_V = x = \theta_V + x \quad \forall x \in V$$

(4) $\forall x \in V \quad \exists\ (-x) \in V$ s.t

$$x + (-x) = \theta_V = (-x) + x$$

(5) $\alpha \in \mathbb{F},\ x \in V \Rightarrow \alpha \cdot x \in V$

(6) $\alpha, \beta \in \mathbb{F},\ x \in V \Rightarrow (\alpha + \beta) x = \alpha x + \beta x$

(7) $\alpha, \beta \in \mathbb{F},\ x \in V \Rightarrow (\alpha\beta) x = \alpha(\beta x)$

(8) $\alpha \in \mathbb{F},\ x, y \in V \Rightarrow \alpha(x+y) = \alpha x + \alpha y$

(9) $1x = x \quad \forall x \in V$

Then we say $\underline{\underline{V}}$ is a Vector space over the field $\underline{\underline{\mathbb{F}}}$ with respect to the addition operation $+$ & scalar multiplication.

The elements of a vector space are called <u>vectors</u>.