Vector Space

V : arbitrary nonempty set

$\mathbb{F}$: field

$+$: addition

$\cdot$: scalar multiplication

Rules for $+, \cdot$

EXAMPLES

(1) $\mathbb{R}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{R} \right\}$

$\mathbb{R}^k$ is a vector space over $\mathbb{R}$

with the laws of + & scalar multiplication as defined earlier.

θ_k : zero element

For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$; $(-x) = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix}$

2) $V = \mathbb{C}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{C} \right\}$

$\mathbb{F} = \mathbb{C}$

V is a vector space over F with the usual laws of + and scalar multiplication.

$$\theta_V = \theta_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$, $(-x) = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix}$

3) $V = \mathbb{F}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{F} \right\}$ $\mathbb{F}$ any field

Then $\mathbb{F}^k$ is a vector space over $\mathbb{F}$ with the usual laws of addition and scalar multiplication

$$\theta_V = \theta_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

For $x \in V$, $(-x) = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix}$

Ex:

$\mathbb{F} = \mathbb{R} \longrightarrow$ we get $\mathbb{R}^k$

$\mathbb{F} = \mathbb{C} \longrightarrow$ we get $\mathbb{C}^k$

$\mathbb{F} = \mathbb{Z}_2 = \{0, 1\}$ $(\mathbb{Z}_2, +, \times)$

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

$(-0) = 0$

$(-1) = 1$

In $(\mathbb{Z}_2, +, \times)$ every element is its own negative

$\underline{k=3}$

$$V = \mathbb{Z}_2^3 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{Z}_2 \right\}$$

$$\mathbb{Z}_2^3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$V = \mathbb{Z}_2^3$ is a vector space over $\mathbb{Z}_2$ with 'usual' laws of <u>addition</u> and <u>scalar</u> <u>multiplication</u>

For ex

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\mathbb{Z}_2^3$ over $\mathbb{Z}_2$ (Vector Space)

V $\qquad$ $\mathbb{F}$

$$\theta_V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{Z}_2^3 \text{ then } (-x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For ex

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 0+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \theta_V$$

4) $V = \mathbb{R}^{m\times n} = \left\{ A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} ; a_{ij} \in \mathbb{R} \right\}$

$\mathbb{F} = \mathbb{R}$

V is a vector space over $\mathbb{F}$
$\mathbb{R}^{m\times n}$ $\mathbb{R}$

with "usual" laws of addition & scalar mult of matrices.

$$\theta_V = 0_{m\times n}$$

$A \in V$ then $(-A) = (-a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$

$\mathbb{R}^{n^2}$: Vect space of all $n \times n$ real matrices over $\mathbb{R}$.

5) $V = \mathbb{C}^{m \times n}$ is a vect space over $\mathbb{F} = \mathbb{C}$

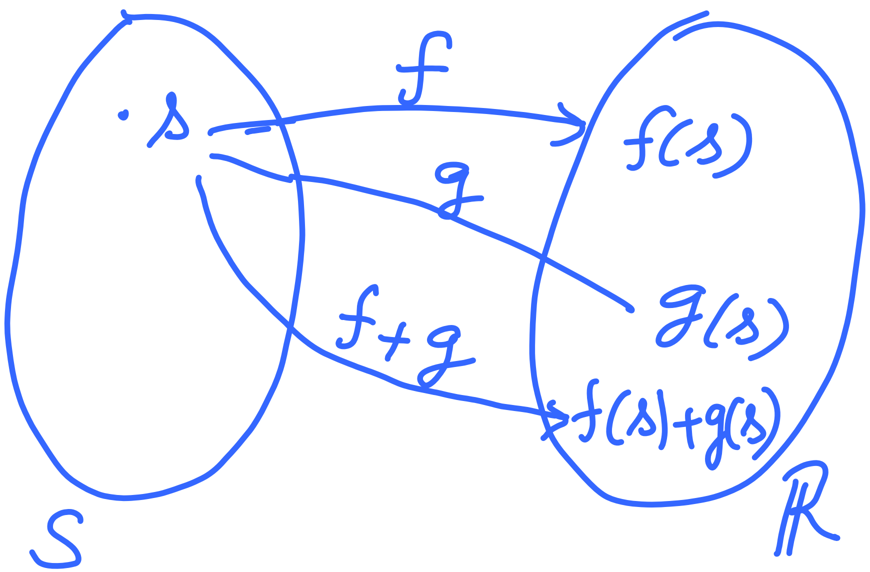
$\theta_V = 0_{m \times n}$ $\quad (-A) = (-a_{ij})$

6) $V = \mathbb{F}^{m \times n}$ is a vect space over $\mathbb{F}$ with usual laws of addition and scalar multiplication

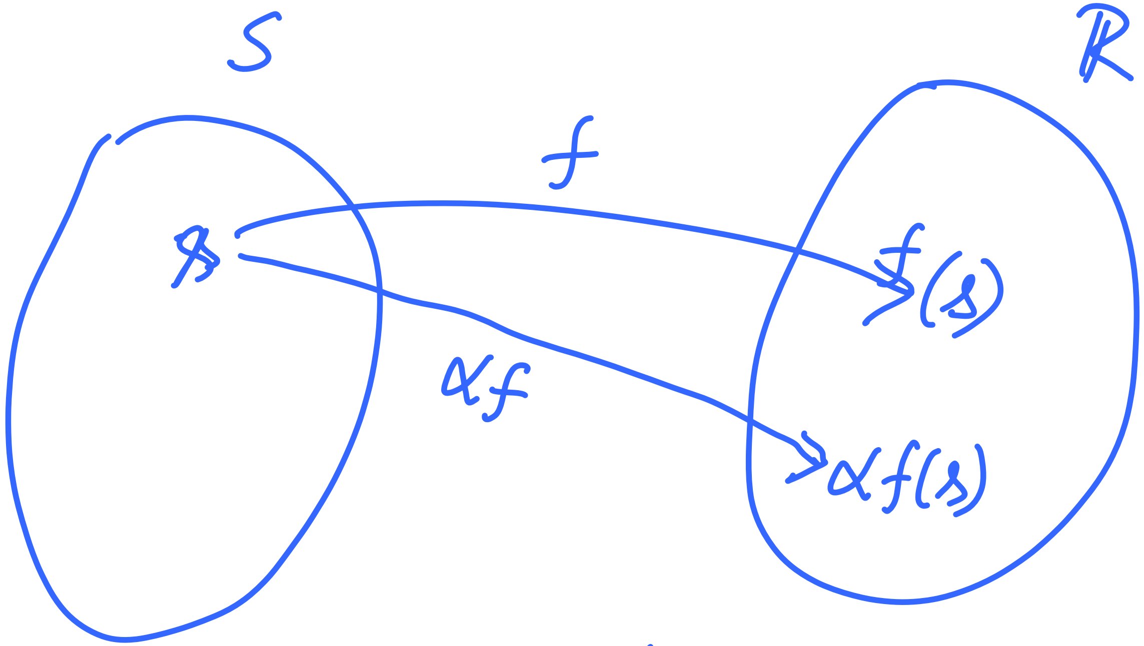
$\theta_V = 0_{m \times n}$ $\quad (-A) = (-a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

$\mathbb{F}^{n^2}$ is the vector space of all $n \times n$ matrices over $\mathbb{F}$

(7)



(Pointwise addition of functions)



$\alpha \in \mathbb{R}$

Scalar multiplication:
(Pointwise Scalar Multiplication)

Let S be any nonempty set

$$\underline{\underline{V}} \overset{F=\mathbb{R}}{\equiv} \{f : S \longrightarrow \mathbb{R}\}$$

$= \mathcal{F}[S; \mathbb{R}]$

$+$: Pointwise add

$f, g, \in \mathcal{F}[S, \mathbb{R}]$;

$$(f+g)(s) \stackrel{def}{=\!=} f(s) + g(s)$$

$\cdot$ Pointwise scalar mult

$\alpha \in \mathbb{R}$, $f \in \mathcal{F}[S; \mathbb{R}]$

$$(\alpha f)(s) = \alpha (f(s))$$

$V = \mathcal{F}[S; \mathbb{R}]$ is a Vector Space over $\mathbb{R}$

$\theta_V : S \longrightarrow \mathbb{R}$ defined as

$$\theta_V(s) = 0 \quad \forall s \in S$$

$f \in \mathcal{F}[S, \mathbb{R}]$ what is $(-f)$

$(-f) : S \longrightarrow \mathbb{R}$ defined as

$(-f)(s) = -(f(s))$

<u>Ex:</u> $S = \{s_1, s_2, \ldots, s_k\}$

$\mathcal{F}[S, \mathbb{R}]$

$f \in \mathcal{F}[S, \mathbb{R}]$

means

$f : S \longrightarrow \mathbb{R}$

$f(s_1), f(s_2), \ldots, f(s_k) \in \mathbb{R}$

We can form

$$\begin{pmatrix} f(s_1) \\ \vdots \\ f(s_k) \end{pmatrix}$$

Basically therefore $\mathcal{F}[S, \mathbb{R}]$ in this case can be "identified" with $\mathbb{R}^k$

Next let

$$S = \{s_1, s_2, s_3, - - - - \}$$

$$\mathcal{F}[S; \mathbb{R}]$$

$$f \in \mathcal{F}[S; \mathbb{R}]$$

$$\Rightarrow f : S \longrightarrow \mathbb{R}$$

We get

$$f(s_1), f(s_2), - - -$$ real number

We get a "sequence" of real numbers

When $S = \{s_1, s_2, - - \}$

$\mathcal{F}[S; \mathbb{R}]$ is basically identified with sequences of real number

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$S = I$, an interval in $\mathbb{R}$

$\mathcal{F}[I;\mathbb{R}] = \{f : I \longrightarrow \mathbb{R}\}$

We may take

$I = [0, 2\pi)$, $I = [0, T)$

$I = (-\infty, \infty)$

Similarly let S be any nonempty set

$\mathcal{F}[S;\mathbb{C}] = \{f : S \longrightarrow \mathbb{C}\}$

Then $\mathcal{F}[S; \mathbb{C}]$ is a vector space over $\mathbb{C}$ with the usual laws of addition & scalar mult.

$$S = \{s_1, s_2, \dots \}$$

$$\mathcal{F}[S, \mathbb{F}] = \{f : S \to \mathbb{F}\}$$

(where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$)

$f \in \mathcal{F}[S; \mathbb{F})$ $\qquad f . S \longrightarrow \mathbb{F}$

$\downarrow$

We get the values

$f(s_1), f(s_2), ---$

$|f(s_1)|, |f(s_2)|, ---$

Add all these

$$\sum_{j=1}^{\infty} |f(s_j)| \text{ may be } \begin{matrix} <\infty \\ =+\infty \end{matrix}$$

$$\ell^1[S;\mathbb{R}] = \left\{ f \in \mathcal{F}[S, \mathbb{F}] : \sum_{j=1}^{\infty} |f(s_j)| < \infty \right\}$$

$$= \ell^1[S, \mathbb{R}]$$

For example if we take

$$f : S \longrightarrow \mathbb{F}$$

defined as

$$f(s_j) = \frac{1}{j^2}$$

Then $\Rightarrow |f(s_j)| = \frac{1}{|j|^2} = \frac{1}{j^2}$

$$\Rightarrow \sum_{j=1}^{\infty} |f(s_j)| = \sum_{j=1}^{\infty} \frac{1}{j^2} \text{ converges } < \infty$$

$\Rightarrow$ This $f \in \ell^1[S, R]$

However if we take

$$f : S \longrightarrow R \text{ as}$$

$$f(s_j) = \frac{1}{j}$$

then $|f(s_j)| = \frac{1}{j}$

$\Rightarrow \sum_{j=1}^{\infty} |f(s_j)| = \sum_{j=1}^{\infty} \frac{1}{j}$ diverges to $+\infty$

$\Rightarrow$ Thus $f \notin \ell^1[S, \mathbb{F}]$

$\ell^1[S, \mathbb{F}]$ is a vect. space over $\mathbb{F}$

Analogously instead of asking

$$\sum_{j=1}^{\infty} |f(s_j)| \text{ to be } < \infty$$

We can look at f for which

$$\sum_{j=1}^{\infty} |f(s_j)|^2 < \infty$$

$$l^2[S, \mathbb{R}] = \left\{ f : S \to \mathbb{R} : \sum_{j=1}^{\infty} |f(s_j)|^2 < \infty \right\}$$

For ex we had if

$$f : S \longrightarrow R \text{ is s.t}$$

$$f(s_j) = \frac{1}{j}$$

Then $f \notin l^1[S; R]$

But now this fcn is s.t

$$\sum_{j=1}^{\infty} |f(s_j)|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

$\therefore$ Thus $f \in \ell^2[S, \mathbb{R}]$

& $f \notin \ell^1[S; \mathbb{R}]$

$\ell^2(S, \mathbb{F})$ is a Vect Space over $\mathbb{F}$

<u>Similarly</u>

$I \subseteq \mathbb{R}$	$I \subseteq \mathbb{R}$
$\mathcal{F}[I; \mathbb{R}]$	$\mathcal{F}[I; \mathbb{C}]$
Vect sp over $\mathbb{R}$	Vect space over $\mathbb{C}$

$S = I$

$$L^1[S;\mathbb{R}] = \left\{ f \in \mathcal{F}[S,\mathbb{R}] : \int_I |f(t)|\,dt < \infty \right\}$$

$$L^2[S;\mathbb{R}] = \left\{ f \in \mathcal{F}[S,\mathbb{R}) : \int_I |f(t)|^2\,dt < \infty \right\}$$