Subspace — 4 important subspaces
of a space $\mathbb{F}^n$ & $\mathbb{F}^m$
Concerning an $m \times n$ matrix

$A \in \mathbb{F}^{m \times n}$

$\mathbb{F}^n$	$\mathbb{F}^m$
Range A^T $= \text{col}(A^T)$ $= \text{Row}(A)$	Range A $= \text{col}(A)$ $= \text{Row } A^T$
Null Sp A	Null Sp A^T

Subspace spanned by a set S

V vect space over a field $\mathbb{F}$

S finite set $u_1, \ldots, u_r$ in V

$\mathcal{L}[S]$: The coll. of all l.c. of S vectors

This is a i) subspace

ii) subspace that contains S

iii) Smallest subspace that contains

What if S is an infinite set?

Consider any finite subset $\hat{S}$ of S

Then we can look at $\mathcal{L}[\hat{S}]$

$\mathcal{F}_S$: The collection of all nonempty finite subsets of S

$\forall\ \hat{S} \in \mathcal{F}_S$ we can define $\mathcal{L}[\hat{S}]$

$$\bigcup_{\hat{S} \in \mathcal{F}_S} \mathcal{L}[\hat{S}] \overset{\text{def}}{=} \mathcal{L}[S]$$

$\mathcal{L}[S]$: The collection of all possible finite l.c. of S vectors

Easy to check that

S is a subspace

S is a subspace containing S

S is the smallest subspace containing S

$\mathcal{L}[S]$ is called the subspace spanned by S

It is clear that the def

$$\mathcal{L}[S] = \bigcup_{\hat{S} \in \mathcal{F}_S} \mathcal{L}[\hat{S}]$$

coincides with earlier def. when S is finite

Linear Independence

Recall if

$S = u_1, \ldots, u_r$ finite set in V

then we say S is l.i. if

$$\alpha_1 u_1 + \ldots + \alpha_r u_r = \theta_V, \quad (\alpha_j \in \mathbb{F})$$

then $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$

Suppose S is infinite set in V

Def: An infinite set S in V is said to be l.i. if every finite subset of S is l.i.

Example: $\mathbb{F}[x] = V$

$$S = \{1, x, x^2, \dots, x^n, \dots\}$$

This S is an infinite set in V.

Is this l.i?

Yes $\because$ every finite set consists only of a finite no. of powers of x

Ex 2 $I = [0, 2\pi]$

$\mathcal{C}[I; \mathbb{R}]$

$$V = \mathcal{C}[I, \mathbb{R}] = \{ f : [0, 2\pi] \to \mathbb{R} \}$$

$$S = \{ \sin(nx) \}_{n=1,2,3,\cdots}$$

This S is a l.i. set in

Why?

Consider any finite subset of S

$$S_1 = \sin(n_1 x), \sin(n_2 x), \cdots, \sin(n_r x)$$

To check if S_1 is l.i.

$$\alpha_1 \sin(n_1 x) + \alpha_2 \sin(n_2 x) + \cdots + \alpha_r \sin(n_r x) = 0$$

$$\Longrightarrow \alpha_1 \sin(n_1 x) \sin(n_j x) + \cdots + \alpha_r \sin(n_r x) \sin(n_j x) = 0$$

(Integrate from 0 to 2π)

Use $\int_0^{2\pi} \sin(kx)\sin(lx)\,dx = 0$ if $k \neq l$
$= \pi$ if $k = l$

$$\alpha_j \pi = 0 \implies \alpha_j = 0 \qquad (1 \leq j \leq n)$$

S_1 is l.i.

Thus every finite subset of S is l.i.

Hence S is l.i

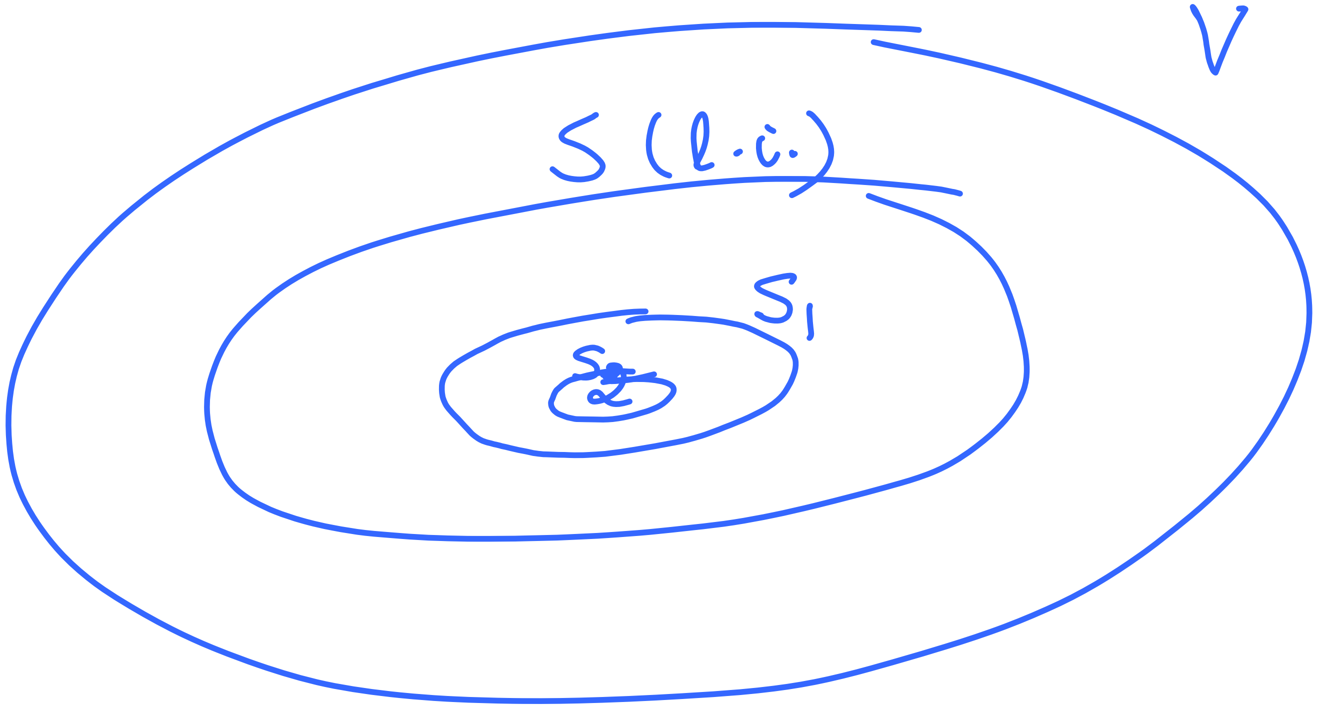
<u>Remark:</u> 1. If S is inf & l.d

– means at least one

finite subset of S must be l.d

SOME PROPERTIES OF LI and LD

(1)



$S \subset V$ S is l.i

$S_1 \subset S$; S_1 nonempty

Is S_1 l.i?

Suppose not

This means S_1 has a finite subset S_2 which is l.d

$\Rightarrow S_2 \subset S_1 \subset S$

& $\therefore S_2$ is a finite l.d. subset of S

$\Rightarrow S$ cannot be l.i.

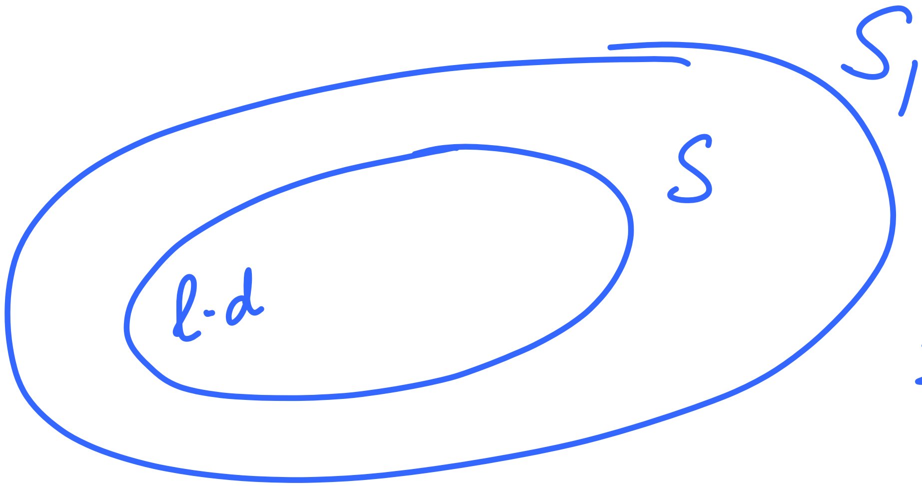
— contradicts

Hence S_1 is also l.i

CONCLUSION

Any Nonempty subset of a l.i. set is l.i

2)



S l-d.

$S_1 \supset S$

$\Rightarrow S_1$ is l-d

Conclusion: Any Superset of a l-d. set is l-d.

3) S l.d

S l.d

$\Rightarrow \exists\, S_1 \subset S$ s.t

S_1 is finite &

S_1 is l d

$$S_1 = u_1, u_2, \dots, u_r$$

Since S_1 is l-d.

$\exists\, \alpha_1, \alpha_2, \dots, \alpha_r$ not all of which are zero

s.t.

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r = \theta_V$$

Let j be the largest index for which $\alpha_j \neq 0$

$\therefore \alpha_{j+1} = 0 = \alpha_{j+2} = - \quad = \alpha_r$

We get:

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j = \theta_V$ & $\alpha_j \neq 0$

$\Rightarrow u_j = \beta_1 u_1 + \cdots + \beta_{j-1} u_{j-1}$

where $\beta_i = -\dfrac{\alpha_i}{\alpha_j} \quad \left(= -\alpha_i \alpha_j^{-1} \right)$

$\Rightarrow u_j$ is a l.c. of finite no. of vectors in S

<u>CONCLUSION</u>

If S is l.d. $\exists$ at least one $u \in S$ s.t u is a l.c. of a finite no. of vectors in S

4) $S = u_1, u_2, \ldots, u_r$ is a finite l.d. set.

$\Rightarrow \exists\ \alpha_1, \ldots, \alpha_r$ not all of which are 0 $\ni$

$$\alpha_1 u_1 + \cdots + \alpha_r u_r = \theta_V$$

Let j be the largest index s.t. $\alpha_j \neq 0$

$$\Rightarrow \alpha_1 u_1 + \cdots + \alpha_j u_j = \theta_V$$

$$\Rightarrow u_j = \beta_1 u_1 + \cdots + \beta_{j-1} u_{j-1}$$

$$(\beta_i = -\alpha_i \alpha_j^{-1})$$

CONCLUSION

If $S = u_1, u_2, \ldots, u_r$ is a finite l.d. set then $\exists$ a vector among them which is a l.c. of all the preceding vectors

5) $S = u_1, u_2, \ldots, u_r$ l.d finite

$\mathcal{L}[S] = (\text{Interested})$

W.l.g. assume all these vectors as nonzero vector

$u_1 \quad u_2 \quad \cdot\!/\!-\!/ \quad u_r$

Scan from the left

Remove the first vector that you get from left which is a l.c. of the previous vectors

Repeat the Process on the remaining subset

After a finite no. of steps we get $S_1 \subset S$ s.t. $\mathcal{L}[S_1] = \mathcal{L}[S]$

& S_1 is l.i.

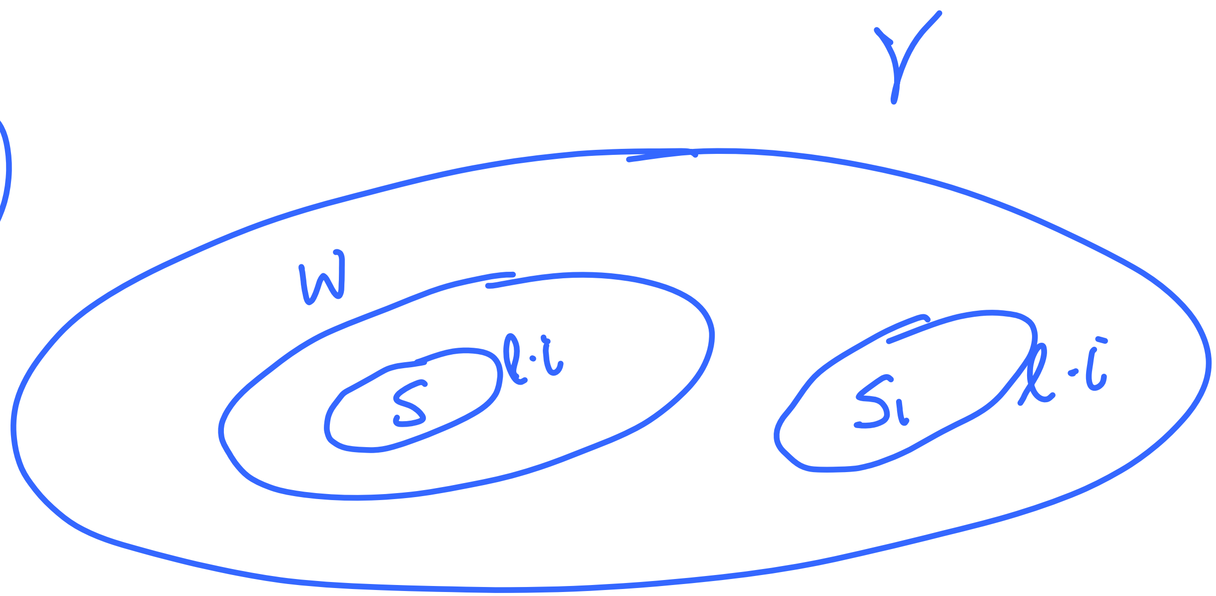
CONCLUSION

If S is a finite l.d. set then $\exists$ a subset S_1 of S

s.t. i) S_1 is l.i. &

ii) $\mathcal{L}[S_1] = \mathcal{L}[S]$

6)



V : Vect Space

W : Subspace of V

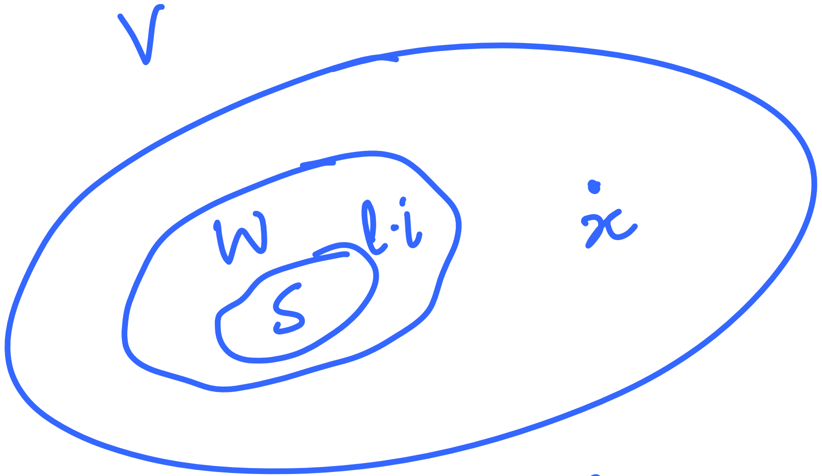
$S \subset W$ is l.i.

$S_1 \subset V \setminus W$ and S_1 l.i.

$\Rightarrow$ $S \cup S_1$ is l.i in V

(Why?)

<u>In particular</u>



$W \subset V$

$W \neq V$

$S \subset W$ l.i.

$x \in V \setminus W$

Can $x = \theta_V$?

No ($\because \theta_V \in W$)

$S_1 = \{x\}$ is l.i.

$\therefore \quad S \cup S_1 = S \cup \{x\}$ is l.i

BASIS