{ Linear Indep
Linear Dep }

Spanning Set

K is a ld set (finite) in a subspace W of a vect sp. V

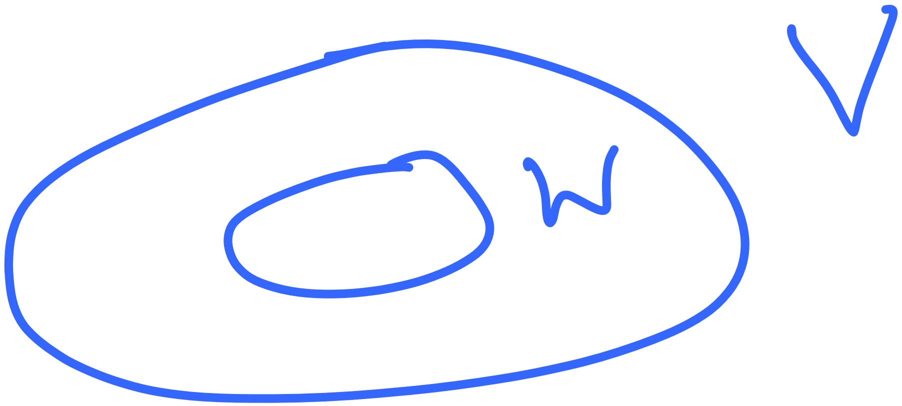
then by our scanning process from the left and get

$$K_1 \subsetneq K$$

s.t $K_1$ is a l-i set

and $\mathcal{L}[K_1] = \mathcal{L}[K]$

**Question** Should every subspace have a spanning set?




$V$ vect sp

$W$ subspace of $V$

$W$ as a subset $W$

What is $\mathcal{L}[W]$?

$$\mathcal{L}[W] = W$$

$\therefore$ We can think of $W$ as a spanning set of $W$.

There is at least one spanning set for $W$.

What is a spanning set.

— Like a Sampling set.

Would like spanning set to be a subset of $W$

A l.d. spanning set is like a "over sampling" set

Hence we would like to have a l.i. spanning set

**Def:** Let $V$ be a Vect Sp. over $\mathbb{F}$

$W$ be a subspace of $V$

Then a subset $\mathcal{B} \subset W$ is called a BASIS for $W$ if

i) l.i. and

ii) $L[\mathcal{B}] = W$ — (i.e. every $w \in W$ can be expressed as a l.c. of finite number of vectors in $\mathcal{B}$

Basically the two requirements for a basis:

i) l.i. & ii) Spanning set

$\therefore$ A subset $S$ of $W$ will fail to be a basis of at least one of the above conditions is not satisfied

**Remark:**

We do not know yet whether a subspace of $V$ will have a basis

(We do not know yet whether $V$ has a basis)

<u>Example:</u>

(1) $\mathbb{F}^3 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_j \in \mathbb{F} \right\}$

$\mathcal{B}: e_1, e_2, e_3$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then clearly $\mathcal{B}$ is a subset of $V$

s.t. i) $\mathcal{B}$ is l.i.

ii) $\mathcal{B}$ spans $V$ i.e. $\mathcal{L}[\mathcal{B}] = V$

$\because x \in V \Rightarrow x \in \mathbb{F}^3 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $x_j \in \mathbb{F}$

$$\Rightarrow x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

Hence $\mathcal{B}$ is a basis for $\mathbb{F}^3$

SIMILARLY for $\mathbb{F}^n$

$$\mathcal{B} = e_1, e_2, \ldots, e_n,$$

where

$$e_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j^{th} \\ \\ \\ \\ \end{array},$$

is a basis for $\mathbb{F}^n$

<u>Ex 2:</u> $V = \mathbb{F}^{m \times n}$

In particular $V = \mathbb{F}^{2 \times 3}$

$$\mathcal{B}: A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$$

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}, \quad A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$$

$$A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then 1) $\mathcal{B}$ is l.i (Easy to check this)

2) $\mathcal{L}[\mathcal{B}] = \mathbb{F}^{2\times 3}$

$$\therefore A \in \mathbb{F}^{2\times 3} \Rightarrow A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}, \quad a, b, c, d, e, f \in \mathbb{F}$$

$\Rightarrow A = a A_{11} + b A_{12} + c A_{13}$
$+ d A_{21} + e A_{22} + f A_{23}$

$\Rightarrow A$ is l.c. of $\mathcal{B}$ vector

$\Rightarrow A \in \mathcal{L}[\mathcal{B}]$

$\therefore \mathcal{B}$ is l.i., $\mathcal{L}[\mathcal{B}] = \mathbb{F}^{2\times 3}$

$\therefore \mathcal{B}$ is a basis for $\mathbb{F}^{2\times 3}$

Analogously if $V = \mathbb{F}^{m \times n}$

$$\mathcal{B} = \{A_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

where

$$A_{ij} = \begin{array}{cc} & \begin{array}{ccc} & j^{th} & \end{array} \\ i \rightarrow & \begin{pmatrix} 0 & & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix}_{m \times n} \end{array}$$

Then $\mathcal{B}$ is a basis for $\mathbb{F}^{m \times n}$

3) $F[x]$ = The set of all polynomials in $x$ with coeffts in $F$.

$$\mathcal{B} = \{ p_0, p_1, p_2, \ldots, p_n, \ldots \}$$

where $p_n(x) = x^n$

1) $\mathcal{B}$ is l.i

2) $p \in V \Rightarrow p \in F[x]$

$\Rightarrow p = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$

$(a_j \in F)$

$\Rightarrow p = a_0 p_0 + a_1 p_1 + \dots + a_k p_k$

$\Rightarrow p \in \mathcal{L}[\mathcal{B}]$

$\therefore$ $\mathcal{B}$ is a basis for $V$.

In this space consider the following subspace

$W = F_e[x] = \{ p \in F[x] : p(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{2n} x^{2n} \}$

$\mathcal{B} = \{ 1, x^2, x^4, \cdots, x^{2n}, \cdots \}$

is a basis for $W$

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Recall Two imp requirements for a subset $S \subset W$ (subspace of $V$)

to be a basis for $W$ are:

1) S must be l.i, and

2) $\mathcal{L}[S] = W$

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## View Basis from a different Point of View

V vect space over F; W subspace of V

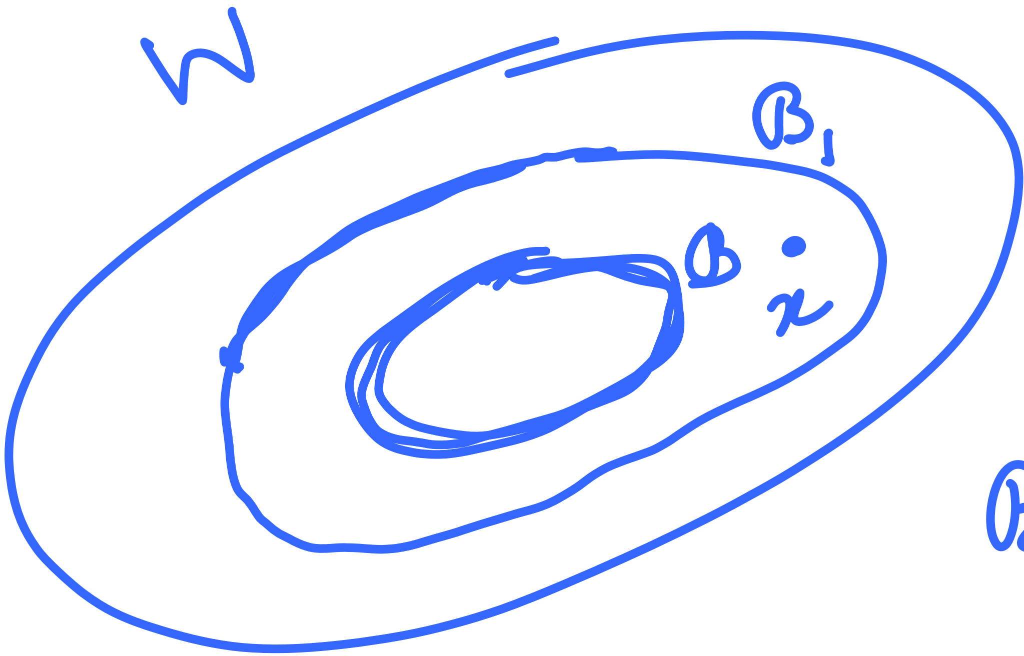
$S \subset W$ is a maximal l.i. set

if 1) l.i and

2) S is not a Proper Subset of any other l.i set

(that is if $S \subsetneq S_1 (\subset W)$

$\Rightarrow S_1$ is l.d.)




$\mathcal{B}$ is a basis for $W$

Let $\mathcal{B}_1 \subset W$

be s.t.

$\mathcal{B} \subsetneq \mathcal{B}_1$

$\exists\, x \in \mathcal{B}_1 \setminus \mathcal{B}$

$x \in W$ (and $\mathcal{B}$ is a basis for $W$)

Hence $x$ can be written as a l.c of a finite number of vectors in $B$ — say $u_1, \ldots, u_r$

$\Rightarrow$ $x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r$

$\Rightarrow$ $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + (-1)x = \theta_V$

$\Rightarrow$ $\underbrace{u_1, u_2, \ldots, u_r,}_{B \text{ in } B_1}\ \underset{\text{in } B_1}{\underset{\downarrow}{x}}$ is a l.d. set

$\Rightarrow$ $u_1, u_2, \ldots, u_r, x$ is a l.d. set in $B_1$

$\Rightarrow$ $B_1$ is l.d

CONCLUSION -

$\mathcal{B}$ is a basis for $W$,

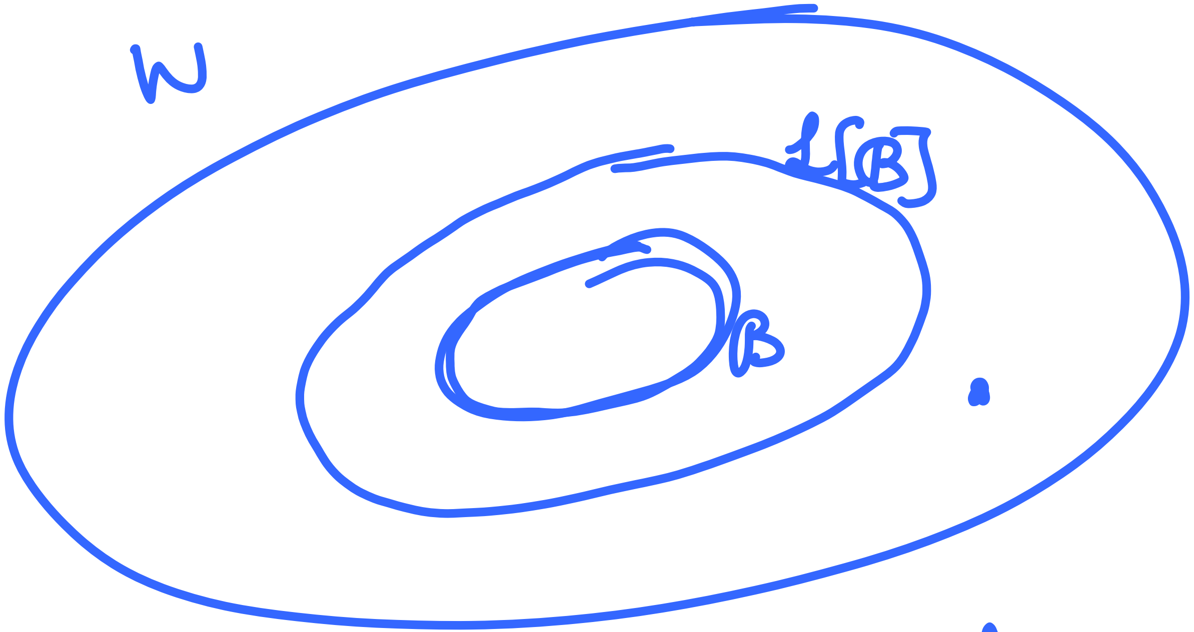
$\mathcal{B} \subsetneq \mathcal{B}_1 \subset W$

$\Rightarrow$ $\mathcal{B}_1$ is l.d

$\Rightarrow$ $\mathcal{B}$ is Maximal l.i. in $W$

$$\boxed{\begin{array}{l} \mathcal{B} \text{ is a basis for } W \\ \Rightarrow \mathcal{B} \text{ is a Max l.i. set in } W \end{array}}$$

Conversely let $\mathcal{B}$ be a maximal l.i set in W.




Claim:

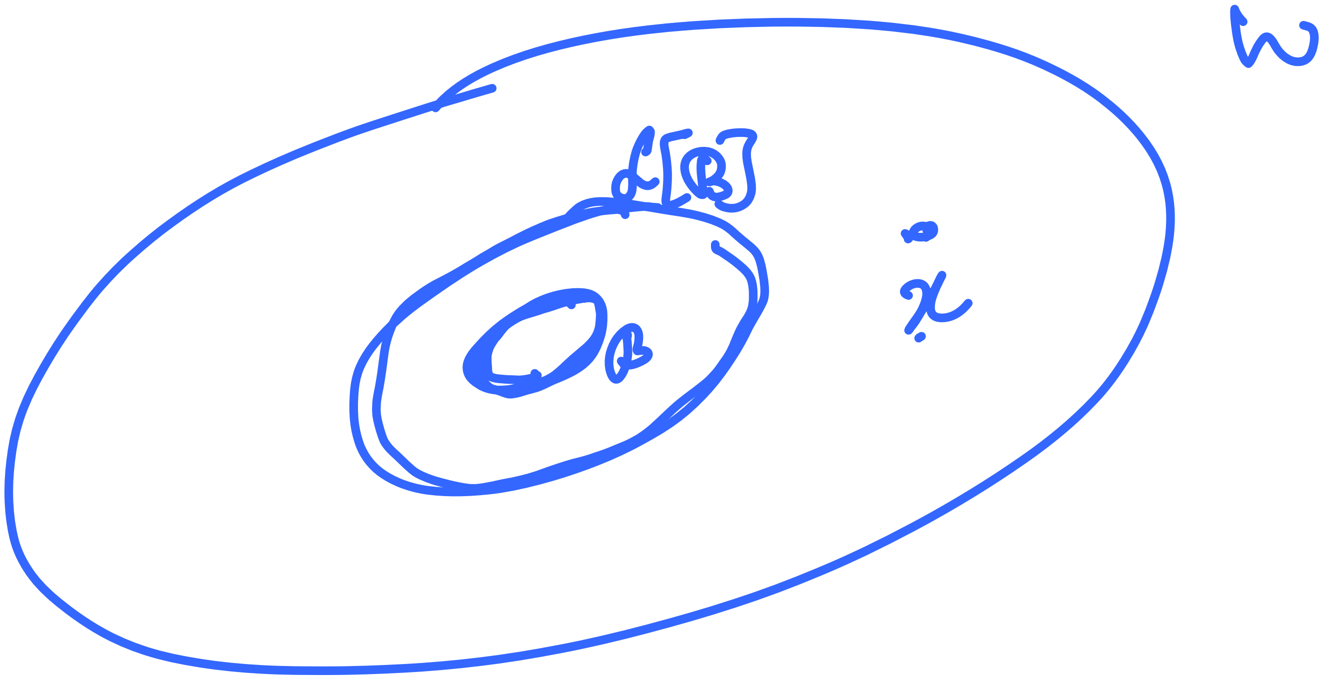
$\mathcal{B}$ is a basis for W

Suppose not

$\therefore$ Since $\mathcal{B}$ is l.i the only way $\mathcal{B}$ can fail to be a basis for W is by $\mathcal{L}[\mathcal{B}] \neq W$

$\therefore$ $\mathcal{L}[\mathcal{B}] \subsetneq W$

$\mathcal{L}[B]$ is a subspace of $V$ }
$B$ is l.i. set in $\mathcal{L}[B]$ }




$\exists\, x \in W \ni x \notin \mathcal{L}[B]$

$\therefore B$ is a l.i. set in $\mathcal{L}[B] \subset W$
& $x$ is outside $\mathcal{L}[B]$ but in $W$

Hence $B \cup \{x\}$ is a l.i. set

$\Rightarrow$ $B$ is a proper subset of the l.i. set $B \cup \{x\}$

$\Rightarrow$ $B$ is not a max l.i. set in $W$

— Contradiction because we started with a max l.i set $B$.

Hence our assumption that $B$ is not a basis is false

$\mathcal{B}$ is Max l.i. set in $W$

$\Longrightarrow$ $\mathcal{B}$ is a basis for $W$

By these two conclusions we get

$\mathcal{B}$ is a Basis for $W$

$\Longleftrightarrow$ $\mathcal{B}$ is a Max l.i. set in $W$

<u>Remark:</u> It is this interpretation of a basis together with what is known as Zorn's lemma that assumes that every vect space has a basis (& ∴ every subspace of a vect. space has a basis)

## Example:

1) $\mathbb{F}^3$

$\mathcal{B}$: $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

1) $\mathcal{B}$ is l.i.

2) Suppose $\mathcal{B}_1$ is any set in $\mathbb{F}^3$

s.t. $\mathcal{B} \subsetneq \mathcal{B}_1$

$\therefore \exists\, x \in \mathcal{B}_1 \setminus \mathcal{B}$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$x = x_1 e_1 + x_2 e_2 + x_3 e_3$

$\Rightarrow$ $e_1, e_2, e_3, x$ l.d. set in $B_1$

$\Rightarrow$ $B_1$ is l.d

$B$ is a Max l.i. set

<u>Example</u>: $\mathbb{F}^{2\times 3}$

$B$: $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$

1) $B$ is l.i

iii) $\mathcal{B}_1 \supsetneq \mathcal{B}$ means $\exists\, A \in \mathcal{B}_1 \setminus \mathcal{B}$

and let $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$

$$A = aA_{11} + bA_{12} + cA_{13} + dA_{21} + eA_{22} + fA_{23}$$

$\Rightarrow A_{11}, \dots, A_{23}, A$ l-d in $\mathcal{B}_1$

$\Rightarrow \mathcal{B}_1$ is l.d.

$\Rightarrow$ $\mathcal{B}$ is Max l.i.

Ex 3. $\mathbb{F}[x]$

$(p_n = x^n)$

$\mathcal{B} = \{1, x, x^2, \ldots, x^n, \ldots\}$

$= \{p_0, p_1, p_2, \ldots, p_n, \ldots\}$

i) $\mathcal{B}$ is l.i in $V$

2) $\mathcal{B} \subsetneq \mathcal{B}_1 \subset V$

$\exists\ p \in \mathcal{B}_1 \setminus \mathcal{B}$

Let $p = a_0 + a_1 x + \ldots + a_k x^k$

$\Rightarrow \quad p = a_0 p_0 + a_1 p_1 + \cdots + a_k p_k$

$\Longrightarrow \quad p_0, p_1, \ldots, p_k, p$ l d in $B_1$

$\Longrightarrow \quad B_1$ is l.d.

Hence $B$ is Max l.i

**BASIS for W** $\begin{cases} \text{l.i. set in } W \\ \text{SPANS } W \end{cases}$

OR

$\{$ max l.i. set in $W$

# Finite dimensional Spaces

Let $W$ be a subspace of $V$ and let $W$ have a finite basis. Then we say $W$ is a finite dimensional subspace.

If $V$ itself has a finite basis we say $V$ is a finite dimensional vector space.

Let $V$ be a finite dimensional vect

Space

This means there is a finite basis

Let it be

$$\mathcal{B} : u_1, \ldots, u_n$$

(i.e. $u_1, \ldots, u_n \in V$
l.i
span $V$)

Suppose $B_1 : v_1, \ldots, v_n, v_{n+1}$

Claim: $B_1$ must be l.d.

Suppose not

Then $B_1$ is l.i.

Look at $S_1 = v_1, u_1, u_2, \ldots, u_n$ l.d

($\because v_1$ is a l.c of $u_1, \ldots, u_n$ since $B$ is a basis)

By scanning from left

we get

$B_1 = v_1, B_1^{(1)}$ spans same space as $S$

where $\mathcal{B}_1^{(1)} \subsetneq \mathcal{B}$

$\therefore \quad \mathcal{L}[S_1] = \mathcal{L}[\mathcal{B}_1] = V$

$\left(\because \mathcal{L}[S_1] = V\right.$ since $u_1, \ldots, u_n$ basis vec are already in $\left. S_1 \right)$

$v_2, v_1, \mathcal{B}_1^{(1)}$

Apply scanning

$\mathcal{B}_2 = v_2, v_1, \mathcal{B}_1^{(2)}$ span $V$

Continue

$\mathcal{B}_r = v_r, v_{r-1}, \ldots, v_1$ span $V$

and $r \leq n$

$\therefore$ $v_r, v_{r-1}, \ldots, v_1$ is l.i
span $V$

$\therefore$ Basis for $V$

$\Rightarrow$ $v_{n+1}$ is a l.c. of $v_r, v_{r-1}, \cdot, v_1$

$\Rightarrow$ $v_1, \ldots, v_n, v_{n+1}$ must be l.d.

— Contradiction

Hence $B_1$ ~~no~~ : $v_1, \ldots, v_{n+1}$
must be l.d.

If $V$ has a basis having $n$ elements then any set having $n+1$ elements is l.d.