Spanning Set of a Subspace W

Interpret as $\longrightarrow$ Sampling set for W

$\longrightarrow$ Optimal Sampling set

$\longrightarrow$ Look for a l.i. spanning set

$\longrightarrow$ Basis for a subspace W
(Max l.i set in W)
(or) l.i. sp. set

Simplest Situation:

W has a finite basis

→ Finite dimensional Space

<u>Def</u>: V vect space over $\mathbb{F}$

W is a subspace of V

We say W is a finite dimensional subspace of V if W has a finite basis

We say V is a finite dimensional vector space (f.d.v.s.) if V has a finite basis

Some Properties

V vector space over $\mathbb{F}$

$W \subset V$ is a subspace of V
and W is finite
dimensional

$\Longrightarrow$ W has a finite basis

say $\mathcal{B} = u_1, u_2, \ldots, u_d$

Consider any subset in W
which has $d+1$ vectors,

$\mathcal{B}_1 = v_1, v_2, \ldots, v_d, v_{d+1}$

We claim $\mathcal{B}_1$ must necessarily be l d

<u>Reason:</u> Suppose $\mathcal{B}_1$ is l.i.

$v_1 \in W$

$\mathcal{B}$ basis for W
& hence $\mathcal{L}[\mathcal{B}] = W$

$\Longrightarrow v_1 \in \mathcal{L}[\mathcal{B}]$

$\Longrightarrow v_1 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_d u_d$

$\Longrightarrow 1 v_1 + (-\alpha_1) u_1 + \cdots + (-\alpha_d) u_d = \theta_V$

$\Longrightarrow v_1, u_1, \ldots, u_d$ must be l.d.

$\Longrightarrow \exists$ a u_i which is a l.c of $v_1, u_1, \ldots, u_{i-1}$

Remove this vector

$v_1, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d$

If this is not l.i. we can

find u_j $(j \geq i+1)$ s.t

u_j is a lc of all precedingones

Remove this vector

Proceding this way

We get after completing this proces

v_1, B_1 (where $B_1 \subset B$)

which is l.i. & spans W

$\therefore$ $v_1, \mathcal{B}_1$ forms a basis for W

Look at

$v_2, v_1, \mathcal{B}_1$ l.d

Use the same procedure as above

We get $\mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{B}$ s.t

$v_2, v_1, \mathcal{B}_2$ is a basis for W

Continuing this process after

r steps, $(r \leq d)$

$v_r, v_{r-1}, \dots, v_1$ is a basis for W

$\Longrightarrow$ Max l.i. set in W

$\Longrightarrow$ Cannot be a proper subset of any l.i. set

— Contradiction
since this is a subset
of the l.i. set B_1

Hence B_1 must be l.d

CONCLUSION

If a subspace W has
a basis consisting of
d vectors then any set in W
consisting of d+1 vectors
must be l.d
Consequently, any set in W
having more than d vectors

must be l d

2) W has a basis $\mathcal{B}$ consisting of d vectors

Suppose $\mathcal{B}_1$ is any other basis for W

$\mathcal{B}_1$ must be l.i

Hence $\mathcal{B}_1$ must have $\leq d$ vectors (since otherwise it will be l d)

CONCLUSION

If W has one basis
consisting of d vectors
then every basis of W
has to be finite and
can contain at most
d vectors

3) W is a finite
dimensional subspace of V
B_1 is a basis having
m vectors

B_2 is a basis having

n vectors

B_1 · basis has m vectors
Any other basis can have
at most m vectors
B_2 must have at most m vectors

$$\boxed{n \le m}$$

B_2 basis has n vectors
Any other basis has at most n vectors
B_1 must have at most n vectors

$$\boxed{m \le n}$$

Hence $m = n$

Conclusion

If W is a finite dimensional subspace of a vector space V then all basis for W will have the same no. of vectors

Definition:

Let W be a f.d subspace of a vector space V.

The number of vectors in any basis is called the DIMENSION of the subspace W

In particular if V is a f.d.v.s then the number of vectors in any basis is called the DIMENSION of V

EXAMPLES

(1) $\mathbb{F}^3$

$\mathcal{B} = e_1, e_2, e_3$ is a basis for $\mathbb{F}^3$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This has 3 vectors in it

Hence $\dim \mathbb{F}^3 = 3$

Let us look at the subspace

$$W = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix} : \alpha, \beta \in \mathbb{F} \right\}$$

$\mathcal{B}$. $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; $u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a basis for W.

$\mathcal{B}$ has two vectors

Hence

$\dim W = 2$

3) $\mathbb{F}^n$ has basis

$\beta: e_1, e_2, \ldots, e_n$

where

$$e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{matrix} \qquad j = 1, 2, \ldots, n$$

β has n vectors in it

$\therefore \dim \mathbb{F}^n$ is n

4) $\mathbb{F}^{2\times 3}$

$\{A_{i,j}\}_{\substack{1\le i\le 2 \\ 1\le j\le 3}}$ is a basis for $\mathbb{F}^{2\times 3}$

$A_{i,j}$ is the 2×3 matrix which has all entries except ij^{th} entry as 0 & ij^{th} entry is 1

For ex

$$A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

There are 6 vectors in $\mathcal{B}$

$\dim \mathbb{F}^{2\times 3} = 2\times 3 = 6$

Similarly

$\dim \mathbb{F}^{m\times n} = m\times n$

$\dim \mathbb{F}^{n\times n} = n^2$

5) $\mathbb{F}[x]$

Suppose $\mathbb{F}[x]$ is f.d. V.S and $\dim = n$

Then there must be a finite basis

say $u_1, u_2, \ldots, u_n$

$\Rightarrow$ Any $n+1$ vectors must be l.d

— Contradiction

since $1, x, x^2, \ldots, x^n$
are $n+1$ l.i. vectors in $\mathbb{F}[x]$

$\therefore$ $\mathbb{F}[x]$ is an infinite dimensional vector space.

Consider
$$W = \mathbb{F}_2[x]$$

Then $\mathcal{B}: 1, x, x^2$ ($\mathcal{L}[\mathcal{B}] = \mathbb{F}_2[x]$
$\mathcal{B}$ l i)

is a basis for W

$\mathcal{B}$ has 3 vectors in it

$\therefore \dim \mathbb{F}_2[x] = 3$

In general if d is any positive integer, then

$$\mathbb{F}_d[x]$$

is a subspace having dim $d+1$

since $1, x, x^2, \ldots, x^d$

is a basis for $\mathbb{F}_d[x]$ having $d+1$ vectors

6) $S = \{s_1, s_2, \ldots, s_k\}$

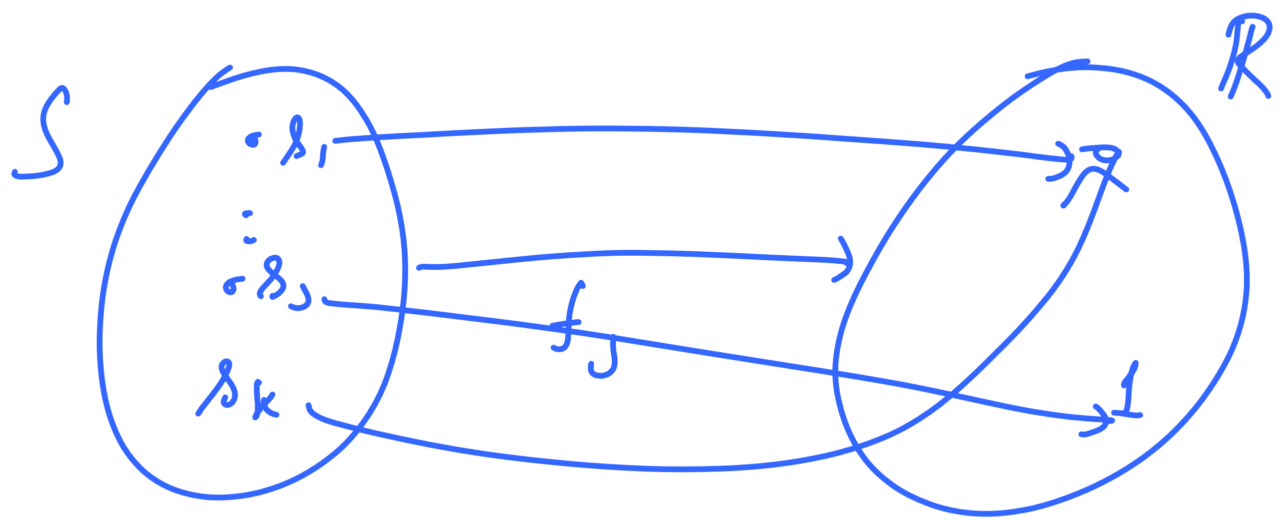
$$\mathcal{F}[S, \mathbb{R}] = \{ f : S \longrightarrow \mathbb{R} \}$$

For each j, $(1 \le j \le k)$ look at

$$f_j : S \longrightarrow \mathbb{R}$$

defined as

$$f_j(t) = \begin{cases} 0 & \text{if } t \neq s_j \\ 1 & \text{if } t = s_j \end{cases}$$



Then look at

$$\{f_j\}_{j=1}^{K} \in \mathcal{F}[S;R]$$

Verify that

i) $\{f_j\}_{j=1}^{K}$ l.i set

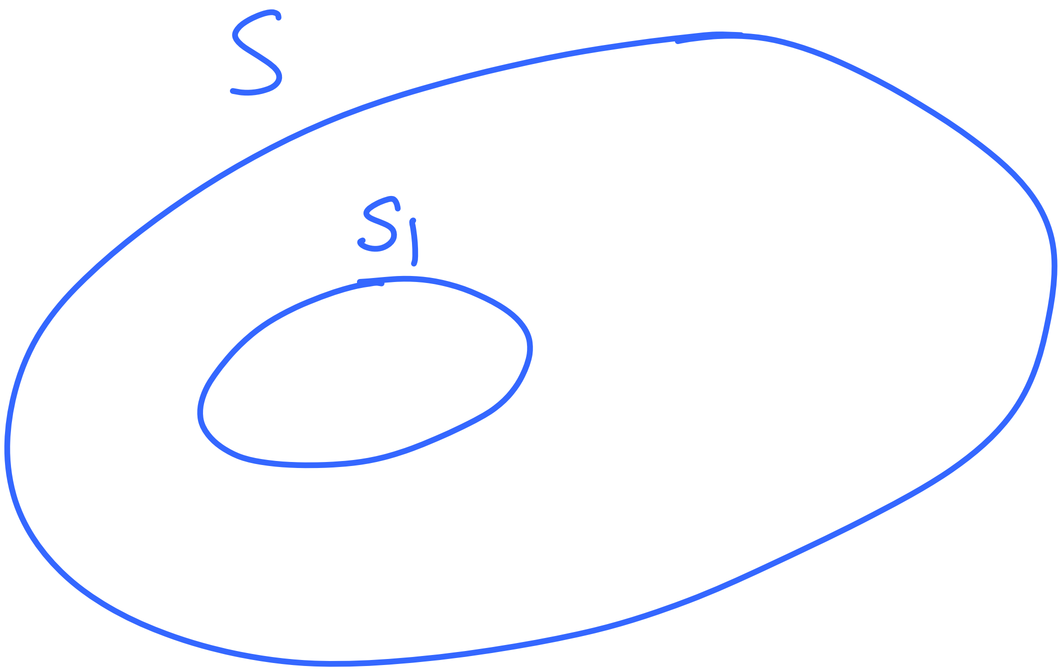
ii) $\{f_j\}_{j=1}^{K}$ is a spanning set for $\mathcal{F}[S;R]$

$\therefore$ Form a basis

This has K vectors Hence

Dimension of $\mathcal{F}[S;R] = K$

(7) S is an infinite set



$S_1 \subset S$ is a finite set

$\mathcal{F}[S, \mathbb{R}]$ inf dim vect. space

$$W = \{ f \in \mathcal{F}[S, \mathbb{R}] : f(s) = 0 \text{ if } s \notin S_1 \}$$

$S_1 = \{s_1, s_2, \ldots, s_k\}$

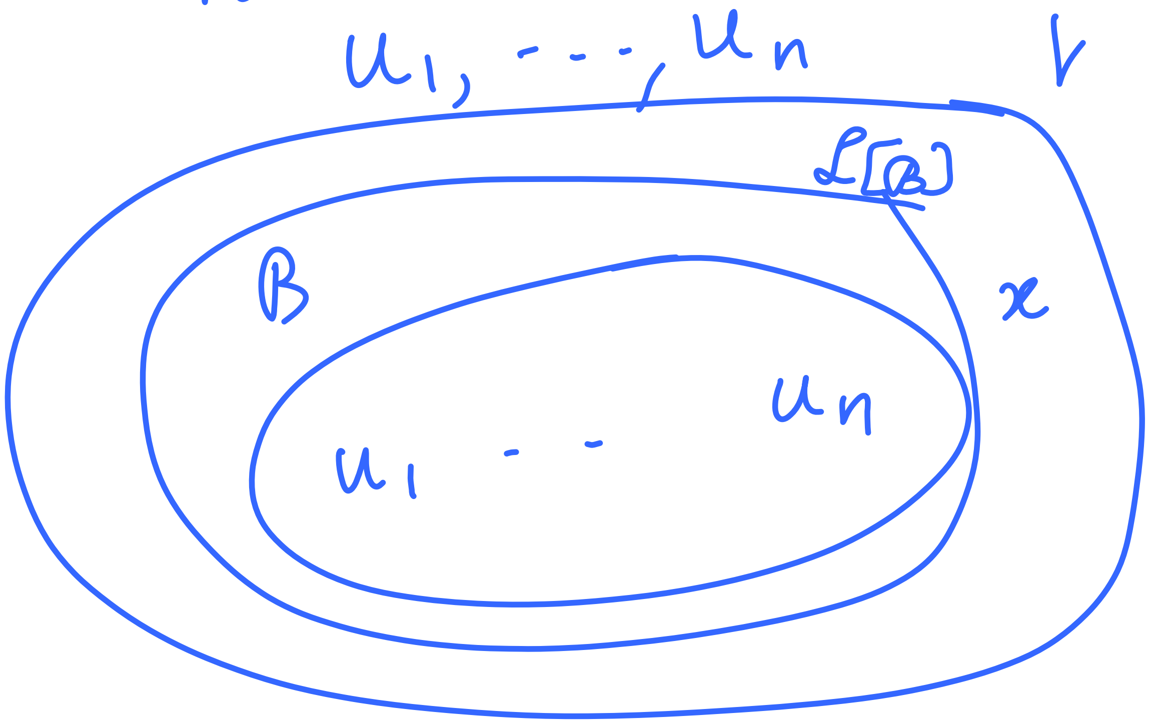
Then use above example to see if W is a f.d. subspace of $\mathcal{F}[S,R]$ and if so whether dimension of W is K.

Some more properties fdvs

Suppose V is a vect sp. of dimension n

1) Any set in V consisting of more than n vectors must be l.d.

2) $\mathcal{B}$ is a l.i set having n vectors $u_1, \ldots, u_n$



How can $\mathcal{B}$ fail to be a basis

if $\mathcal{L}[\mathcal{B}] \neq V$

$\Rightarrow \exists\, x \in V \setminus \mathcal{L}[\mathcal{B}]$

$\Rightarrow \{x\} \cup \mathcal{B}$ is l.i

$\Rightarrow$ ($\mathcal{B}$ is not a max l.i. set)

$\mathcal{B}$ is a set having $n+1$ vect & l.i

— Contradiction

$\therefore$ B is a basis

$\text{Dim } V = n \Rightarrow$ Any n l.i. vectors form a basis