V vector space over $\mathbb{F}$

$W \subset V$ is a f.d. subspace of V

(1) If W has a basis having d vectors then any set in W having more than d vectors is l.d.

(2) All basis for W must be finite and all basis have the same <u>number of vectors</u>

This leads to the notion

<u>Dimension</u> : # vectors in a basis

(Number of)

V vector space over F

$W \subset V$ subspace

$\dim W = d$

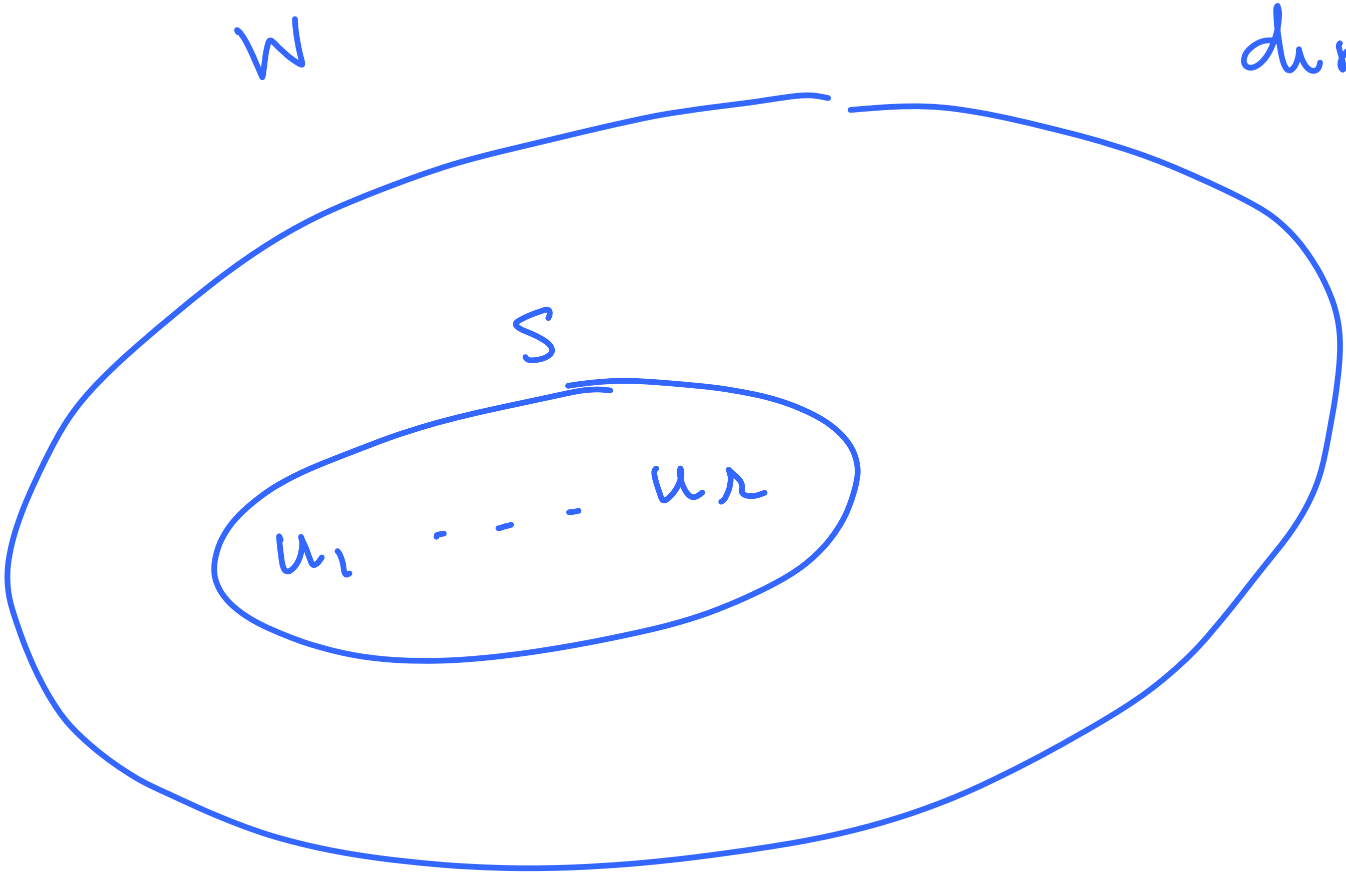
1) Any set in W having more than d vectors must be l.d

2) Any l.i. set in W having d vectors <u>must</u> necessarily be a basis for W

More Properties of f.d. subspaces

3) $W \subset V$ $\dim W = d$

$S \subset W$ has r vectors & is l.i.



$\dim W = d$

$r \leq d$ since S is l.i. (and any set having $> d$ vector in W is l.d)

CASE 1: $r = d$

Then S is a l.i. set,
has now d vectors

$\Longrightarrow$ S is a basis for W

CASE 2 $r < d$

S cannot be a basis for W
since it has less than d vectors

The only way S can fail
to be a basis for W is by

$$\mathcal{L}[S] \neq W$$

$\exists\, v_1 \in W \setminus \mathcal{L}[S]$

Hence $S_1 = u_1, u_2, \dots, u_r, v_1$ is l.i. W

If $r+1 = d$ then we have d l.i. vectors in W & hence a basis for W

If $r+1 < d$, then $\mathcal{L}[S_1] \neq W$

$\exists\, v_2 \in W \setminus \mathcal{L}[S_1] \ni$

$u_1, \ldots, u_r, v_1, v_2$ is li in W

Continuing this process $d-r$ times we get vectors $v_1, v_2, \ldots, v_{d-r}$ s.t

$u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_{d-r}$

is li in W & hence this forms a basis for W (since there are d lc vectors)

CONCLUSION

V vect sp over $\mathbb{F}$

$W \subset V$ subspace $\dim W = d$

Any l.i. set in W is

Either a basis for W

OR it can be "extended"

s.t. it is part of a basis

In particular, if V is a f.d.v.s

then any l.i. set in V is

Either a Basis for V

OR can be extended to be a

part of a basis.

Example $V = \mathbb{F}^3$

$$W = \left\{ x \in \mathbb{F}^3 ; x = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix} : \alpha, \beta \in \mathbb{F} \right\}$$

We have seen that

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

is a basis for W

$$\dim W = 2$$

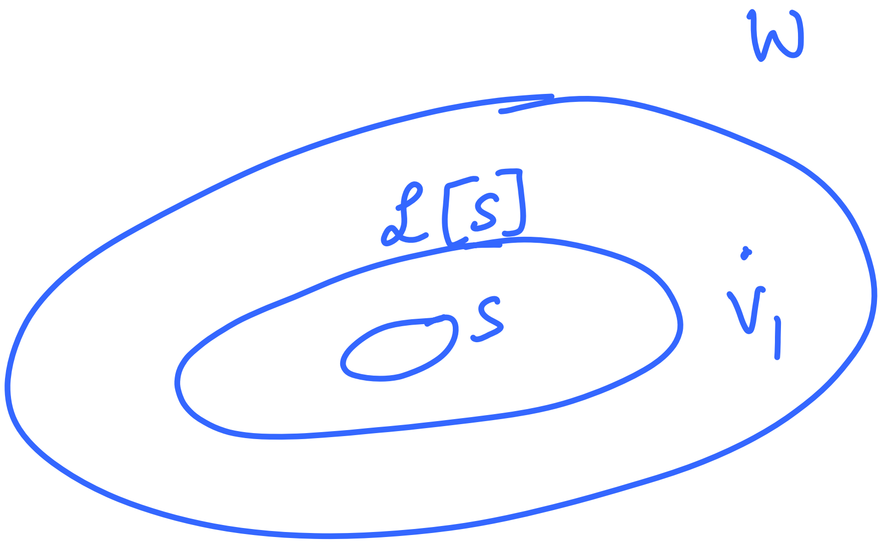
$S = v_1, v_2$

where $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

S is l.i, in W

$r = 2 = d \quad \therefore S$ is a basis

$S = u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ l.i. $\quad r = 1 < 2 = d$



$\mathcal{L}[S]$ vectors are of the form

$\begin{pmatrix} a \\ a \\ 2a \end{pmatrix}, \; a \in \mathbb{F}$

Want $v_1 \in W \setminus \mathcal{L}[S]$

v_1 must be of the form

$$\begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix}$$

And not of the form

$$\begin{pmatrix} a \\ a \\ 2a \end{pmatrix}$$

So we have to choose $\alpha \neq \beta$

For ex. we can choose $\alpha = 1$, $\beta = 2$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

u_1, v_1 is l.i.

This forms a basis.

<u>Note</u>: The choice of v_1 is not unique

(∵ we could have chosen any vector in $W \setminus \mathcal{L}[S]$ as v_1)

For ex take $\alpha = 1$ $\beta = -1$

Get $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

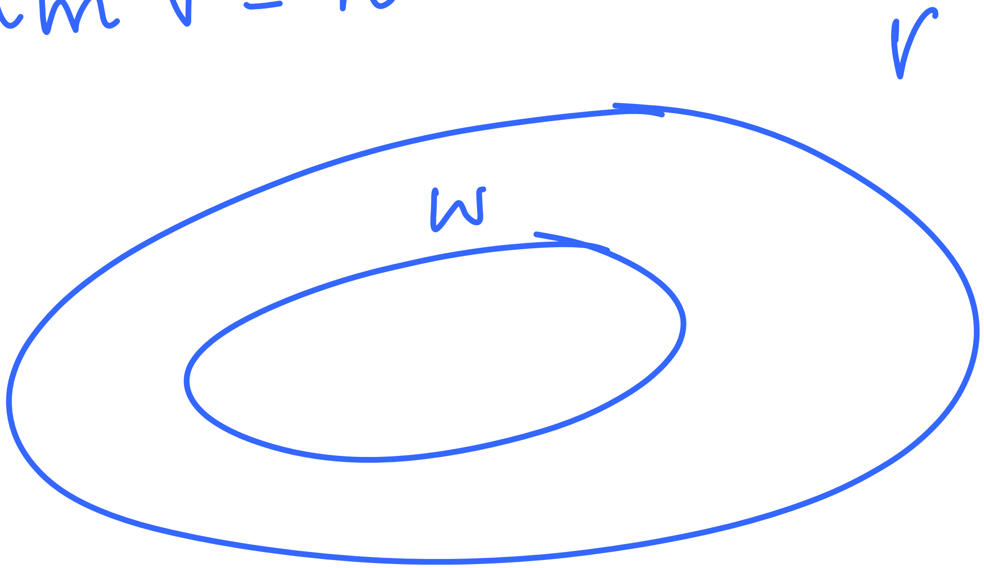
u_1, v_1 is also a basis for W

Thus the "extension" of a l.i. set in W to be a basis for W

is <u>NOT</u> <u>UNIQUE</u>

V vector space over $\mathbb{F}$

$\dim V = n$



$W \subset V$ subspace of V

Since any set of vectors

having more than n vect.
is ld, any basis for W
can have at most n
vectors

Hence $\dim W \leq n$

If $\dim W = n$

Then W has a basis
$u_1, u_2, \ldots, u_n$

There are n li vectors in V
& $\dim V = n$

$\Longrightarrow u_1, \ldots, u_n$ basis for V

Hence $S = u_1, \ldots, u_n$ is s.t.

$\mathcal{L}[S] = W$ $\because$ S is basis for W

& $\mathcal{L}[S] = V$ $\because$ S is basis for V

We get $W = V$

<u>Conclude</u>. $(\dim V = n)$

Any subspace of V is.

Either V

or has dim $< \dim V$

Role of a Basis

V vector space of dim n over $\mathbb{F}$

Ordered basis

A basis for V in which the vectors are arranged in a fixed order is called an ordered basis.

Ex: $\mathbb{F}^3$ $\mathcal{B} = \{e_1, e_2, e_3\}$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathcal{B}_1 = \{e_3, e_1, e_2\}$$

$\mathcal{B} = \mathcal{B}_1$ These are same basis

However, as ordered basis $\mathcal{B}$ & $\mathcal{B}_1$ are different.

V fdvs

$\dim V = n$

$\mathcal{B} = u_1, u_2, \ldots, u_n$ an ordered basis for V (o b)

$\Rightarrow \mathcal{L}[\mathcal{B}] = V$

$x \in V \Longrightarrow x \in \mathcal{L}[\mathcal{B}]$

$\Longrightarrow$ x is a l.c. of $\mathcal{B}$ vectors

$\Longrightarrow \exists\, x_1, x_2, \ldots, x_n \in \mathbb{F}$ s.t

$$x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$$

Now we shall use the fact that $\mathcal{B}$ is l.i to establish that the above representation of $x \in V$ is <u>unique</u>

That is if possible, let

$$x = x_1' u_1 + x_2' u_2 + \cdots + x_n' u_n$$

$$\Longrightarrow \theta_V = (x_1 - x_1')u_1 + \cdots + (x_n - x_n')u_n$$

$$\Longrightarrow \left.\begin{aligned} x_1 - x_1' &= 0 \\ x_2 - x_2' &= 0 \\ &\vdots \\ x_n - x_n' &= 0 \end{aligned}\right\}$$ Since $u_1, \dots, u_n$ l.i

$$\Longrightarrow x_1 = x_1', \; x_2 = x_2', \; \dots, \; x_n = x_n'$$

CONCLUSION

If $B = u_1, \dots, u_n$ an ob for V then every $x \in V$ has a unique representation

$$x = x_1 u_1 + x_2 u_2 + \dots + x_n u_n$$

(where $x_j \in F$) as a l.c. of

the basis vectors

Thus we can think of x as being made of these n scalars

$x_1, x_2, \ldots, x_n \in \mathbb{F}$

thru' this $\mathcal{B}$

Call x_i as the i^{th} coordinate (component) of x w.r.t the ordered basis $\mathcal{B}$.

Starting from x using the o.b. $\mathcal{B}$ we now get,

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$$

V over $\mathbb{F}$

$\mathcal{B}$ o.b

$\mathbb{F}^n$

$$x \xmapsto{\mathcal{B}} [x]_{\mathcal{B}}$$

(Encode $x \in V$ as $[x]_{\mathcal{B}} \in \mathbb{F}^n$)

EXAMPLES

1 $\quad V = \mathbb{F}^3$

$$\mathcal{B} = e_1, e_2, e_3$$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$$x \in \mathbb{F}^3 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

$$\Rightarrow [x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\hat{\mathcal{B}} = u_1, \quad u_2, \quad u_3$$

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x \in \mathbb{F}^3 \implies x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\implies x = x_2 u_1 + x_3 u_2 + x_1 u_3$$

$$[x]_{\hat{\mathcal{B}}} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

$$B_1 = v_1, \; v_2, \; v_3$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$x \in \mathbb{F}^3 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x = \frac{x_1 + x_2 + x_3}{2} v_1 + \frac{x_1 - x_2 - x_3}{2} v_2 + \frac{-x_1 + x_2 - x_3}{2} v_3$$

$$[x]_{\beta_1} = \begin{pmatrix} (x_1 + x_2 + x_3)/2 \\ (x_1 - x_2 - x_3)/2 \\ (-x_1 + x_2 - x_3)/2 \end{pmatrix}$$