V f.d.v.s. over F

$\dim V = n$ $\qquad \mathcal{B} = u_1, \dots, u_n$ o.b. for V

$x \in V \implies x_1 u_1 + \dots + x_n u_n$

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

x_i = i^{th} coordinate (component) of x w.r.t $\mathcal{B}$

$$\left.\begin{array}{l} \mathcal{B} = u_1, u_2, \dots, u_n \\ \mathcal{B}' = u_1', u_2', \dots, u_n' \end{array}\right\} \text{ o.b for } V$$

$$\begin{tikzcd}
x \in V \arrow[r, "\mathcal{B}"] \arrow[dr, "\mathcal{B}'"'] & {[x]_{\mathcal{B}} \in \mathbb{F}^n} \\
& {[x]_{\mathcal{B}'} \in \mathbb{F}^n}
\end{tikzcd}$$

Since $[x]_{\mathcal{B}}$ & $[x]_{\mathcal{B}'}$ represent the same vector x we expect them to be related in some way

What is this relationship?

Obviously this relationship will depend on the relationship between $\mathcal{B}$ & $\mathcal{B}'$.

How do we find this relationship?

$\mathcal{B} = u_1, \ldots, u_n$

$\mathcal{B}' = u_1', u_2', \ldots, u_n'$

$$x \in V \xrightarrow{\mathcal{B}'} [x]_{\mathcal{B}'} \in \mathbb{F}^n$$

$$u_1 \in V \xrightarrow{\mathcal{B}'} [u_1]_{\mathcal{B}'} = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}$$

$$u_2 \in V \xrightarrow{\mathcal{B}'} [u_2]_{\mathcal{B}'} = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} \in \mathbb{F}^n$$

$$u_j \in V \xrightarrow{\mathcal{B}'} [u_j]_{\mathcal{B}'} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \in \mathbb{F}^n$$

Therefore we get n vectors in $\mathbb{F}^n$,

$[u_1]_{\mathcal{B}'}$, $[u_2]_{\mathcal{B}'}$, $- - , [u_j]_{\mathcal{B}'} , - - - , [u_n]_{\mathcal{B}'}$

Construct a $n \times n$ matrix $\in \mathbb{F}^{n \times n}$ as follows:

$$\begin{pmatrix} [u_1]_{\mathcal{B}'} & [u_2]_{\mathcal{B}'} & - - & [u_j]_{\mathcal{B}'} & - - & [u_n]_{\mathcal{B}'} \end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & & & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nn} \end{pmatrix}$$

$$= [B]_{\beta'}$$

$$x \in V \implies x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$$

$$\implies [x]_\beta = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

We also have

$$x = \sum_{j=1}^{n} x_j u_j$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} b_{ij} u_i' \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij} x_j \right) u_i'$$

$$= \sum_{i=1}^{n} \alpha_i u_i'$$

$$\Longrightarrow [x]_{\mathcal{B}'} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

But $x = x_1' u_1' + \cdots + x_n' u_n'$

& hence

$$[x]_{\beta'} = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} b_{1j} x_j \\ \sum_{j=1}^{n} b_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} b_{nj} x_j \end{pmatrix}$$

$$= [\beta]_{\beta,} [x]_{\beta}$$

$$\Rightarrow \quad [x]_{\mathcal{B}'} = [\mathcal{B}]_{\mathcal{B}'} [x]_{\mathcal{B}}$$

Interchange roles of $\mathcal{B}$ & $\mathcal{B}'$ we get

$$[x]_{\mathcal{B}} = [\mathcal{B}']_{\mathcal{B}} [x]_{\mathcal{B}'}$$

$$[x]_{\mathcal{B}} = [\mathcal{B}']_{\mathcal{B}} [\mathcal{B}]_{\mathcal{B}'} [x]_{\mathcal{B}}$$

(for every $x \in V$)

Let $[\mathcal{B}']_{\mathcal{B}} [\mathcal{B}]_{\mathcal{B}'} = K$

$$[x]_\beta = K\,[x]_\beta \quad \text{for all } x \in V$$

<u>Exercise</u>: Take $x = u_1$

Conclude 1^{st} col of K is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

In general take $x = u_j$

& conclude j^{th} col of K is

$$\begin{array}{c} \\ \\ \\ j^{th} \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

& hence $K = I_n$

i.e $[B']_B [B]_{B'} = I_n$

So $[B']_B$ & $[B]_{B'}$ are inverses

of each other

$$[x]_{B'} = [B]_{B'} [x]_B$$

$$[x]_B = [B']_B [x]_{B'}$$

Example: $V = \mathbb{F}^3$

$\mathcal{B}$: e_1, e_2, e_3

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}; \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{F}^3$ then $[x]_{\mathcal{B}} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$\mathcal{B}'$: u_1, u_2, u_3

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \parallel & \parallel & \parallel \\ \begin{pmatrix}1\\1\\0\end{pmatrix} & \begin{pmatrix}1\\0\\-1\end{pmatrix} & \begin{pmatrix}0\\1\\-1\end{pmatrix}\end{array}$$

$$[x]_{\beta'} = \begin{pmatrix} (x_1 + x_2 + x_3)/2 \\ (x_1 - x_2 - x_3)/2 \\ (-x_1 + x_2 - x_3)/2 \end{pmatrix}$$

$$e_1 = \frac{1}{2} u_1 + \frac{1}{2} u_2 - \frac{1}{2} u_3$$

$$e_2 = \frac{1}{2} u_1 - \frac{1}{2} u_2 + \frac{1}{2} u_3$$

$$e_3 = \frac{1}{2} u_1 - \frac{1}{2} u_2 - \frac{1}{2} u_3$$

$$[B]_{B'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1/2 \\ \frac{1}{2} & -\frac{1}{2} & -1/2 \\ -\frac{1}{2} & \frac{1}{2} & -1/2 \end{pmatrix}$$

Verify $[x]_{B'} = [B]_{B'} [x]_B$

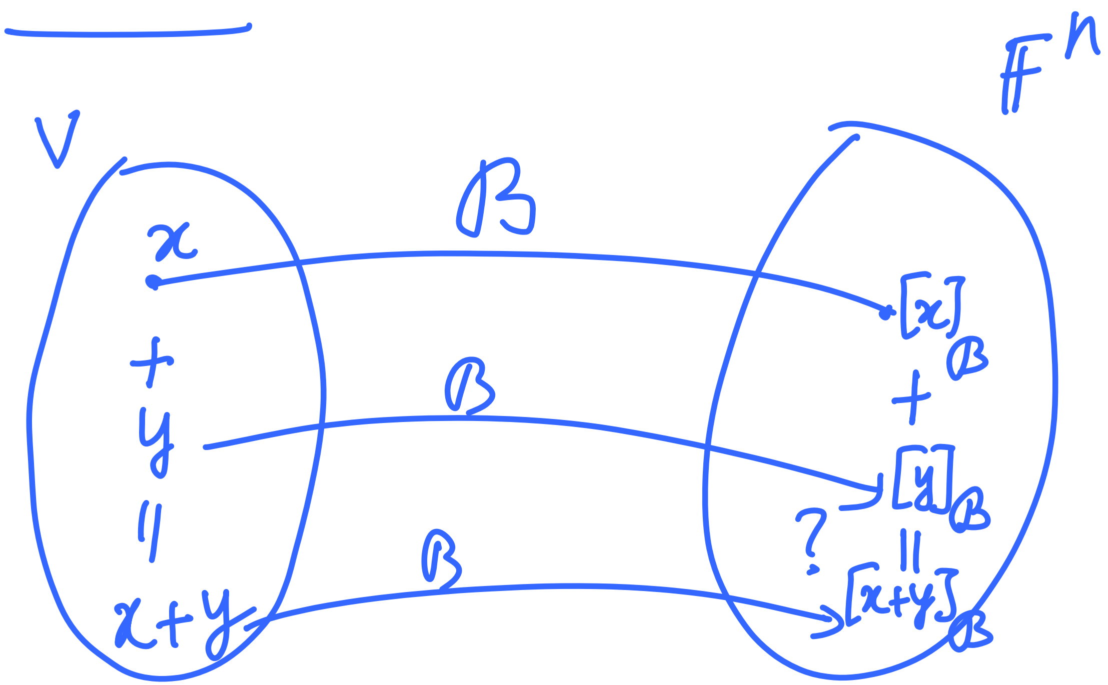
$$\begin{pmatrix} (x_1+x_2+x_3)/2 \\ (x_1-x_2-x_3)/2 \\ (-x_1+x_2-x_3)/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$[\mathcal{B}']_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

<u>Check</u>: $[\mathcal{B}']_{\mathcal{B}} \, [\mathcal{B}]_{\mathcal{B}'} = I_3$

Closer Look at the Representation

$[x]_{\mathcal{B}}$



$$x \xmapsto{\ \mathcal{B}\ } [x]_{\mathcal{B}}$$

$$y \xmapsto{\ \mathcal{B}\ } [y]_{\mathcal{B}}$$

How good is this encoding?

$$x \longmapsto [x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x = x_1 u_1 + \cdots + x_n u_n$$

$$(\mathcal{B} = u_1, \ldots, u_n)$$

$$y \longmapsto [y]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad y = y_1 u_1 + \cdots + y_n u_n$$

$$x + y = (x_1 + y_1) u_1 + (x_2 + y_2) u_2 + \cdots + (x_n + y_n) u_n$$

$$\Longrightarrow$$

$$[x+y]_{\mathcal{B}} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= [x]_{\mathcal{B}} + [y]_{\mathcal{B}}$$

The 'encoding' or the identification of $x \in V$ with $[x]_{\mathcal{B}} \in \mathbb{F}^n$ thru' $\mathcal{B}$ preserves addition

Similarly

$$x \longmapsto [x]_\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Rightarrow x = x_1 u_1 + \cdots + x_n u_n$$

$$\Rightarrow \alpha x = (\alpha x_1) u_1 + \cdots + (\alpha x_n) u_n$$

$\forall \alpha \in \mathbb{F}$

$$\Rightarrow [\alpha x]_\beta = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \alpha [x]_\beta$$

$\Longrightarrow$

$$[\alpha x]_{\mathcal{B}} = \alpha [x]_{\mathcal{B}}$$

The identification of $x \in V$ with $[x]_{\mathcal{B}} \in \mathbb{F}^n$ thru $\mathcal{B}$ preserves scalar multiplication

Thus $x \xmapsto{\mathcal{B}} [x]_{\mathcal{B}}$

preserves addition & scalar multiplication — the two basic operations in a Vector Space.

This leads us to the notion of linear transformation

<u>Def</u>: Let V and W be any two vector spaces over a field F.

Then a map

$$T : V \longrightarrow W$$

which is s.t.

1) $T(x+y) = T(x) + T(y)$
$\forall\, x, y \in V$

and

2) $T(\alpha x) = \alpha\, T(x)$ $\quad \forall\, \alpha \in F$
$\forall\, x \in V$

then we say T is a l.t. (linear transformation)
from V to W

In particular if $V = W$ then a l.t from V to V is called a LINEAR OPERATOR on V

EXAMPLES

Ex 1:

V an n dim vect space over F

$\mathcal{B}$ an ordered basis for V

Define $T_{\mathcal{B}} : V \longrightarrow F^n$

as $T_{\mathcal{B}}(x) = [x]_{\mathcal{B}}$

We have seen

$$T_{\mathcal{B}}(x+y) = [x]_{\mathcal{B}} + [y]_{\mathcal{B}}$$
$$= T_{\mathcal{B}}(x) + T_{\mathcal{B}}(y)$$

& $T_\beta(\alpha x) = [\alpha x]_\beta = \alpha [x]_\beta$

$= \alpha T_\beta(x)$

Hence T_β is a l-t from V to $\mathbb{F}^n$

Thus every ordered basis in an n dimensional vector space V over $\mathbb{F}$ produces a l.t. T_β from V to $\mathbb{F}^n$

<u>Ex 2</u>: $V = \mathbb{F}^3$ $W = \mathbb{F}^2$

Define $T : V \longrightarrow W$

as follows: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$T(x) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ x_1 + 2x_2 - x_3 \end{pmatrix}$$

$$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

$$T(x+y) = \begin{pmatrix} 2(x_1+y_1) - (x_2+y_2) + (x_3+y_3) \\ (x_1+y_1) + 2(x_2+y_2) - (x_3+y_3) \end{pmatrix}$$

$$= T(x) + T(y)$$

Similarly $T(\alpha x) = \alpha T(x)$

So T is a l-t from $\mathbb{F}^3$ to $\mathbb{F}^2$

Similarly

$$T : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$$

defined as

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix}$$

is a l-t from $\mathbb{F}^3$ to $\mathbb{F}^2$

$$T(x) = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= Ax$$

<u>Conclusion:</u>

Any 2×3 matrix $A \in \mathbb{F}^{2\times 3}$ produces a l.t

$$T_A : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$$

as

$$T_A(x) = Ax$$

Generalization

If A is any $m \times n$ matrix in $\mathbb{F}^{m \times n}$ then it produces a l.t

$$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$

as

$$T_A(x) = Ax$$